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Semicontraction and Synchronization of Kuramoto–Sakaguchi Oscillator Networks

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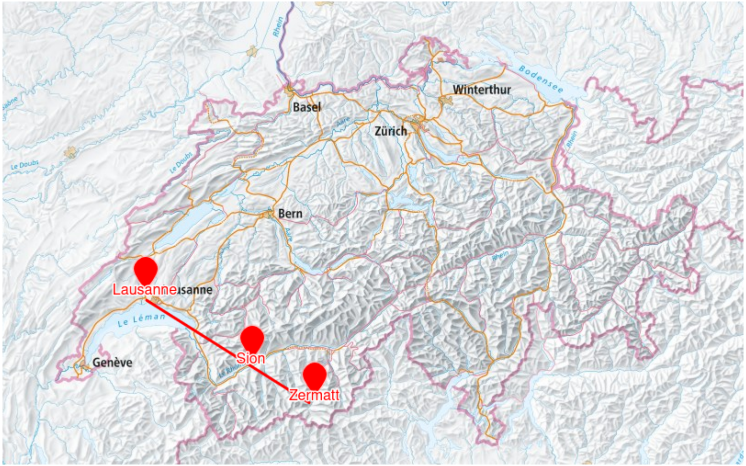
Where is HES-SO Valais-Wallis?



Prequel

Robin

- Working on synchronization
- Available to travel

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Francesco

- Working on contraction theory
- Available to host and supervise

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The Kuramoto-Sakaguchi model

$$\dot{x}_i = \omega_i - \sum_{j=1}^{n} a_{ij}\left[\sin(x_i - x_j - \varphi) + \sin\varphi\right] = f_i(\mathbf{x}), \qquad i \in \{1, \dots, n\}$$
$$= \omega_i - \sum_{j=1}^{n} c_{ij}\sin(x_i - x_j) + s_{ij}[1 - \cos(x_i - x_j)],$$

where:

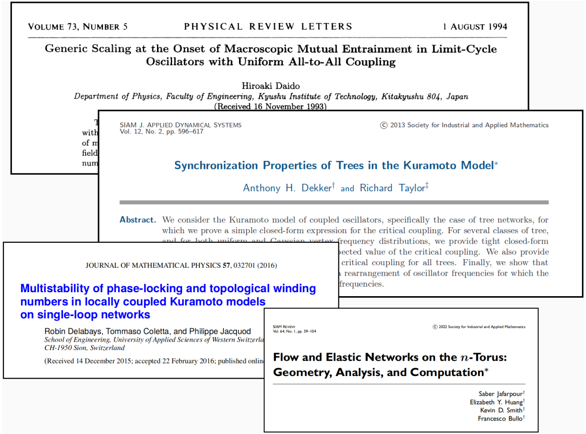
- $x_i \in \mathbb{S}^1 \simeq (-\pi, \pi]$;
- $\omega_i \in \mathbb{R}$;
- $a_{ij} \in \mathbb{R}_{\geq 0}$;
- $\varphi \in (-\pi/2, \pi/2)$;
- $c_{ij} = a_{ij}\cos(\varphi)$;
- $s_{ij} = a_{ij}\sin(\varphi)$.

How many stable synchronous states?

Consider $\omega_i = 0$ *for now.*

What we know:

- Trees: uniqueness;
- Complete graphs: uniqueness;
- Cycles: multistability;
- Complex: hard...



VOLUME 73, NUMBER 5 PHYSICAL REVIEW LETTERS 1 AUGUST 1994

Generic Scaling at the Onset of Macroscopic Mutual Entrainment in Limit-Cycle Oscillators with Uniform All-to-All Coupling

Hiroaki Daido

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Synchronization Properties of Trees in the Kuramoto Model*

Anthony H. Dekker† and Richard Taylor‡

Abstract. We consider the Kuramoto model of coupled oscillators, specifically the case of tree networks, for which we prove a simple closed-form expression for the critical coupling. For several classes of tree, ... frequency distributions, we provide tight closed-form ... pected value of the critical coupling. We also provide ... critical coupling for all trees. Finally, we show that ... a rearrangement of oscillator frequencies for which the ... frequencies.

JOURNAL OF MATHEMATICAL PHYSICS 57, 032701 (2016)

Multistability of phase-locking and topological winding numbers in locally coupled Kuramoto models on single-loop networks

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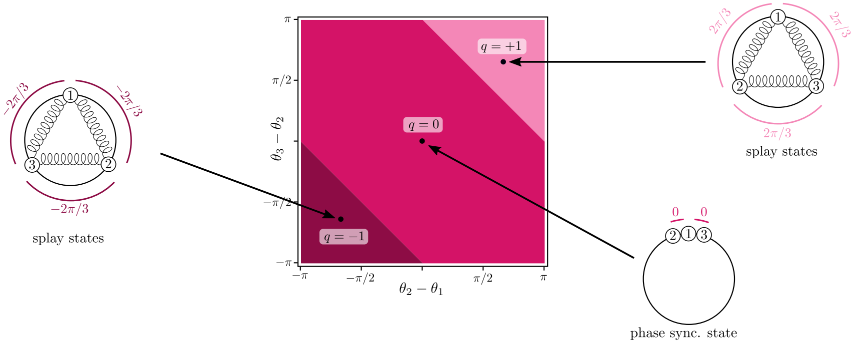
Flow and Elastic Networks on the n-Torus: Geometry, Analysis, and Computation*

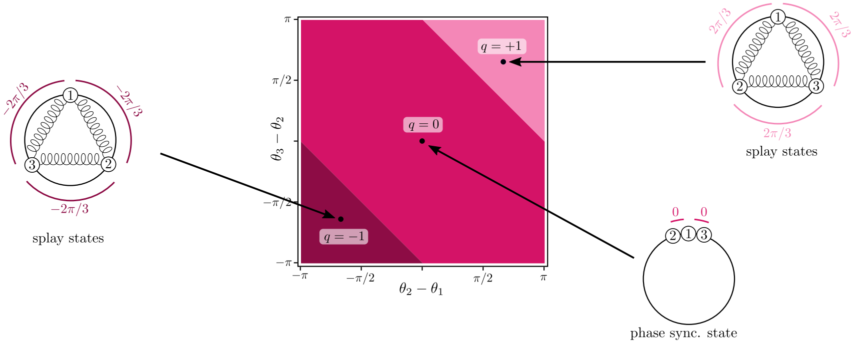
Saber Jafarpour†
Elizabeth Y. Huang†
Kevin D. Smith†
Francesco Bullo†

Multi...



$-2\pi/3$
$-2\pi/3$
$-2\pi/3$
splay states
$q = +1$
$q = 0$
$q = -1$
$\theta_3 - \theta_2$
$\theta_2 - \theta_1$
π
$\pi/2$
$-\pi/2$
$-\pi$
$2\pi/3$
$2\pi/3$
$2\pi/3$
splay states
0
0
phase sync. state

Multi...



The winding cells: $\Omega(u) = \{\mathbf{x} \in \mathbb{T}^n \mid q(\mathbf{x}) = u\}$.

...stability

For all connected i and j:

$$|x_i - x_j| < \pi/2 - \varphi \quad \Longrightarrow \quad \mathrm{Re}[\lambda_i(Df)] \leq 0$$
$$\Longleftrightarrow \quad \text{stability}$$

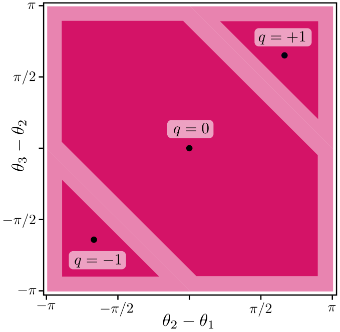
...stability

For all connected i and j:

$$|x_i - x_j| < \pi/2 - \varphi \quad \implies \quad \mathrm{Re}[\lambda_i(Df)] \leq 0$$
$$\iff \quad \text{stability}$$

The cohesive set:

$$\Delta(\gamma) = \{\mathbf{x} \in \mathbb{R}^n \mid \|x_i - x_j\| < \gamma\,,\ i \sim j\}$$



How about contraction?

How about contraction?

$$\dot{x}_i = \omega_i - \sum_{j=1}^{n} c_{ij} \sin(x_i - x_j) - \sum_{j=1}^{n} s_{ij}[1 - \cos(x_i - x_j)] = \omega_i + f^{\mathrm{o}}(\mathbf{x}) + f^{\mathrm{e}}(\mathbf{x}) \,.$$

Invariant along $\mathrm{span}(\mathbf{1}_n)$:

- Dynamics of $\mathbf{x} + \alpha\mathbf{1}_n$ is independent of of α;
- $\lambda_1(Df) = 0$.

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Invariant along $\mathrm{span}(\mathbf{1}_n)$:

- Dynamics of $\mathbf{x} + \alpha \mathbf{1}_n$ is independent of of α;
- $\lambda_1(Df) = 0$.

Recall the *disagreement seminorm*

$$\|x\|_{2,\Pi_n} = \|\Pi_n x\|_2 \,, \qquad \Pi_n = \mathrm{Id} - n^{-1} \mathbf{1}_n \mathbf{1}_n^\top \,.$$

Let $\mu_{2,\Pi}$ denote the induced log-seminorm.

Odd part

$$f_i^{\mathrm{o}}(\mathbf{x}) = -\sum_{j=1}^{n} c_{ij} \sin(x_i - x_j)$$

$$[Df^{\mathrm{o}}(\mathbf{x})]_{ij} = \begin{cases} c_{ij}\cos(x_i - x_j)\,, & \text{if } i \neq j\,, \\ -\sum_k c_{ik}\cos(x_i - x_k)\,, & \text{if } i = j\,. \end{cases}$$

For any $\mathbf{x} \in \mathbb{R}^n$:

- Real eigenvalues: $\lambda_1(Df^{\mathrm{o}}) = 0$ and $\lambda_2(Df^{\mathrm{o}}) \geq \cdots \geq \lambda_n(Df^{\mathrm{o}})$;
- Orthonormal basis of eigenvectors: $v_1, v_2, \dots, v_n$;
- $\ker(\Pi_n) \subseteq \ker(Df^{\mathrm{o}})$ and $\Pi_n v_i = v_i$ for $i \geq 2$.

Odd part (bis)

The Jacobian Df^{o} being a Laplacian matrix, we get

$$\mu_{2,\Pi}(Df^{\mathrm{o}}) = \lambda_2(Df^{\mathrm{o}})\,.$$

F. Bullo, *Contraction Theory for Dynamical Systems, ed. 1.1*, Kindle Direct Publishing (2023). https://fbullo.github.io/ctds

RD and F. Bullo, *Semicontraction and Synchronization of Kuramoto-Sakaguchi Oscillator Networks*, IEEE Control Syst. Lett. **7** (2023).

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The Jacobian Df^{o} being a Laplacian matrix, we get

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Furthermore, by Weyl's inequality, for $\mathbf{x} \in \Delta(\gamma)$,

$$c_{ij} \cos(\gamma) \leq c_{ij} \cos(x_i - x_j) \leq c_{ij} \qquad \Longrightarrow \qquad \lambda_2(Df^{\mathrm{o}}) \leq -\cos(\varphi) \cos(\gamma) \lambda_2(L_{\mathcal{G}}) \,.$$

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Lemma 1: $\mu_{2,\Pi}(Df^{\mathrm{o}}) \leq -\cos(\varphi)\cos(\gamma)\lambda_2(L_{\mathcal{G}})$

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Even part

$$f_i^{\mathrm{e}}(\mathbf{x}) = -\sum_{j=1}^{n} s_{ij}[1 - \cos(x_i - x_j)]$$

$$[Df^{\mathrm{e}}(\mathbf{x})]_{ij} = \begin{cases} s_{ij}\sin(x_i - x_j)\,, & \text{if } i \neq j\,, \\ -\sum_k s_{ik}\sin(x_i - x_k)\,, & \text{if } i = j\,. \end{cases}$$

For any $\boldsymbol{\theta} \in \mathbb{R}^n$,

- $\lambda_1(Df^{\mathrm{e}}) = 0$;
- Off-diagonal is **skew-symmetric**.

Even part (bis)

Let $R = [v_2 \cdots v_n]^\top \in \mathbb{R}^{(n-1)\times n}$, then $\Pi_n = R^\top R$.
We can then compute

$$\mu_{2,\Pi}(Df^{\mathrm{e}}) = \mu_2(RDf^{\mathrm{e}}R^\dagger) = \lambda_{\max}\left(R[\mathrm{diag}(Df^{\mathrm{e}})]R^\top\right) \leq \max(\mathrm{diag}(Df^{\mathrm{e}}))\,.$$

Therefore, for any $\mathbf{x} \in \Delta(\gamma)$,

$$[Df^{\mathrm{e}}]_{ii} = -\sum_{j=1}^{n} c_{ij}\sin(x_i - x_j) \leq \sin(\varphi)\sin(\gamma)\mathrm{deg}_i\,.$$

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Lemma 2: $\mu_{2,\Pi}(Df^{\mathrm{e}}) \leq \sin(\varphi)\sin(\gamma)d_{\max}(\mathcal{G})$

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Altogether

Consider the Kuramoto-Sakaguchi model on $\Delta(\gamma)$.

$$\begin{aligned}\mu_{2,\Pi}(Df) &\leq \mu_{2,\Pi}(Df^{\mathrm{o}}) + \mu_{2,\Pi}(Df^{\mathrm{e}}) \\ &\leq -\cos(\varphi)\cos(\gamma)\lambda_2(L_{\mathcal{G}}) + \sin(\varphi)\sin(\gamma)d_{\mathsf{max}}(\mathcal{G})\end{aligned}$$

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Theorem 1

If, $\gamma < \bar{\gamma} = \arctan\left(\frac{\lambda_2}{d_{\max}\tan(\varphi)}\right)$, then the Kuramoto-Sakaguchi model is **strongly infinitesimally semicontracting on** $\Delta(\gamma)$.

RD and F. Bullo, *Semicontraction and Synchronization of Kuramoto-Sakaguchi Oscillator Networks*, IEEE Control Syst. Lett. **7** (2023).

Convexity and invariance?

One more issue: $\Delta(\gamma)$ is not convex!

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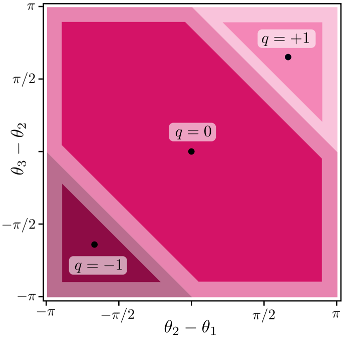
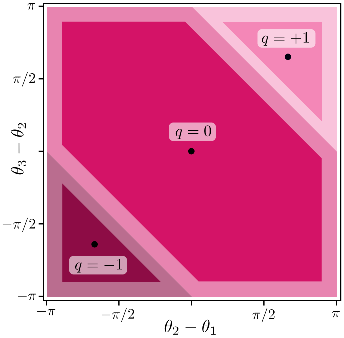


Convexity and invariance?

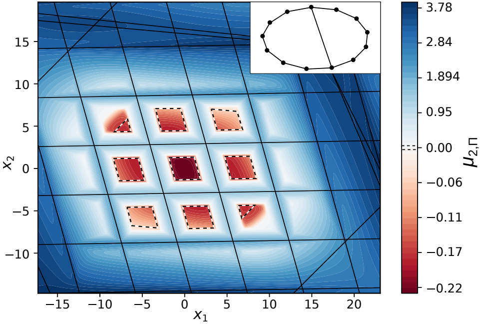
One more issue: $\Delta(\gamma)$ is not convex!

The set $\Gamma_{u,\gamma} = \Omega(u) \cap \Delta(\gamma)$ is convex, but maybe not invariant.

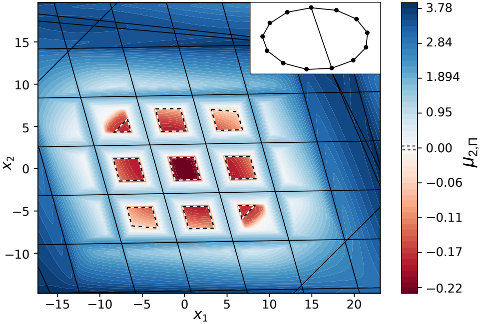
Theorem 2
Let $0 \leq \gamma < \bar{\gamma}$ and $\mathbf{u} \in \mathbb{Z}^c$. There is at most one synchronous state of the Kuramoto-Sakaguchi model in each γ-cohesive winding cell.



Conclusion

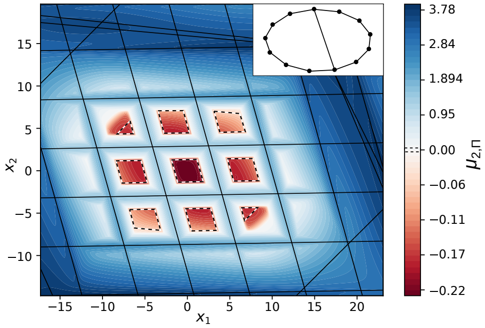

15
10
5
0
−5
−10
x_2
−15
−10
−5
0
5
10
15
20
x_1
3.78
2.84
1.894
0.95
0.00
−0.06
−0.11
−0.17
−0.22
$\mu_{2,\Pi}$

Conclusion



- We showed "at most uniqueness" in $\Gamma(u, \gamma)$;
- The bound on γ involves all relevant ingredients;
- We leveraged various subtlties of contraction theory.

Conclusion



- We showed "at most uniqueness" in $\Gamma(u, \gamma)$;
- The bound on γ involves all relevant ingredients;
- We leveraged various subtlties of contraction theory.

Thank you!

Proof of Lemma 1

We can compute

$$\mu_{2,\Pi}(Df^{\circ}) = \min\{b \mid \Pi Df^{\circ} + Df^{\circ\top}\Pi \preceq 2b\Pi\} = \min\{b \mid Df^{\circ} - b\Pi \preceq 0\}$$

For any $x \in \mathbb{R}^n$,

$$x^{\top}(Df^{\circ} - b\Pi)x = \sum_{i\geq 2}(\mathbf{v}_i^{\top}x)^2[\lambda_i(Df^{\circ}) - b] \leq [\lambda_2(Df^{\circ}) - b]\sum_{i\geq 2}(\mathbf{v}_i^{\top}x)^2 .$$

Therefore,

$$\mu_{2,\Pi}(Df^{\circ}) = \lambda_2(Df^{\circ})$$

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Proof of Lemma 2

Let $R = [v_2 \cdots v_n]^\top \in \mathbb{R}^{(n-1)\times n}$, then $\Pi_n = R^\top R$.
We can then compute

$$\begin{aligned}\mu_\Pi(Df^{\mathrm{e}}) &= \mu_2(RDf^{\mathrm{e}}R^\dagger) = \mu_2(RDf^{\mathrm{e}}R^\top)\\ &= \lambda_{\max}\left(R\frac{Df^{\mathrm{e}} + Df^{\mathrm{e}\top}}{2}R^\top\right)\\ &= \lambda_{\max}\left(R[\mathrm{diag}(Df^{\mathrm{e}})]R^\top\right) \leq \max(\mathrm{diag}(Df^{\mathrm{e}}))\,.\end{aligned}$$

where we used the properties of interlacing eigenvalues (see Cor. 4.3.37 in Horn & Johnson).

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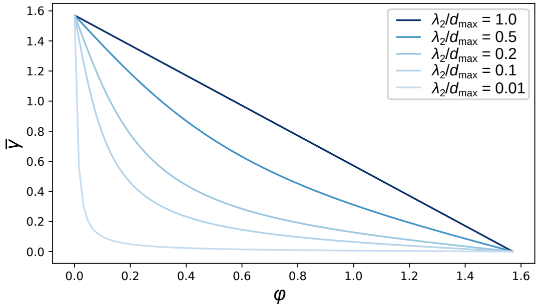
Cohesiveness bound



$$\bar{\gamma} = \arctan\left(\frac{\lambda_2}{d_{\max}\tan(\varphi)}\right)$$