UC SANTA BARBARA

From undirected to directed diffusive networks of dynamical agents

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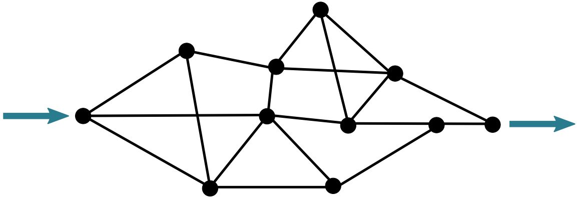
Outline

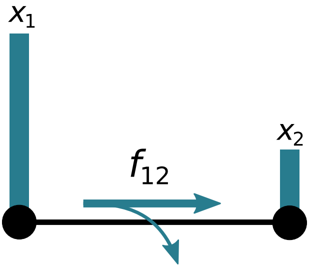
- Introduction
- Power Flows and the Kuramoto model
- Main properties
- Synchronization and Multistability

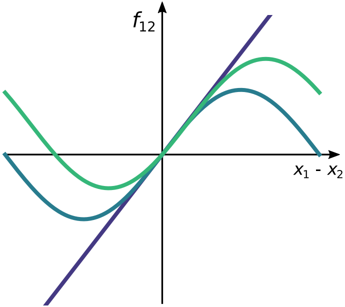
Diffusive networks

How is a commodity transmitted over a network?



Pairwise interaction





$a_{ij} \neq a_{ji}$ **or** $h(x) \neq -h(-x)$

$$f_{ij} = a_{ij} h(x_i - x_j)$$
$$f_{ji} = a_{ji} h(x_j - x_i)$$

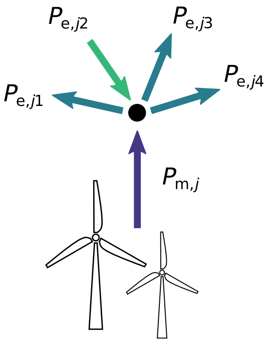
Active power flows:

$$P_{ij} = B_{ij} \sin(\theta_i - \theta_j - \phi)$$

Power flows and the Kuramoto-Sakaguchi model

The power flow equations

- Voltage: $V_j e^{i\theta_j}$.
- Power: $P_j + iQ_j$.
- Admittance: $G_{jk} + iB_{jk}$.
- Electrical power flow: $P_{\mathrm{e},jk}$.



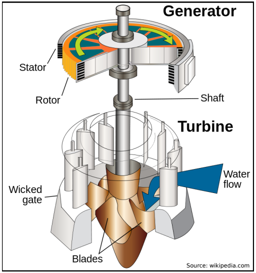
$$P_j = \sum_k V_j V_k \left[B_{jk} \sin(\theta_j - \theta_k) + G_{jk} \cos(\theta_j - \theta_k)\right] ,$$

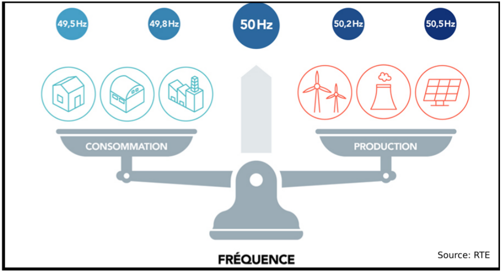
$$Q_j = \sum_k V_j V_k \left[G_{jk} \sin(\theta_j - \theta_k) - B_{jk} \cos(\theta_j - \theta_k)\right] .$$

J. Machowski, J. W. Bialek, and J. R. Bumby, *Power System Dynamics*, 2nd ed. (Wiley, Chichester, U.K, 2008).

The swing equations

$$m_j\ddot{\theta}_j + d_j\dot{\theta}_j = P_{\mathrm{m},j} - P_{\mathrm{e},j}$$
$$= P_j - \sum_k B_{jk}\sin(\theta_j - \theta_k) + G_{jk}\cos(\theta_j - \theta_k)$$

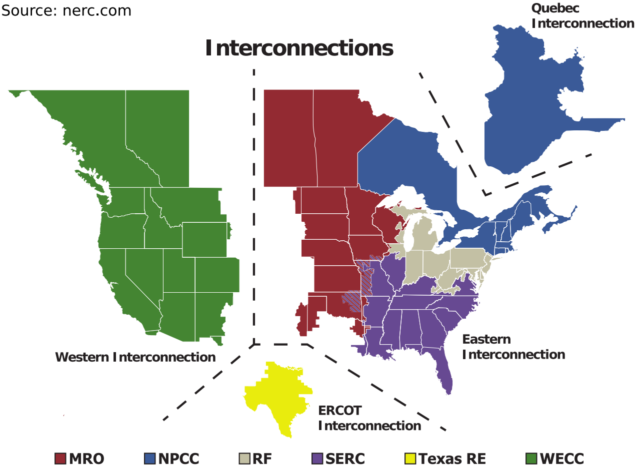




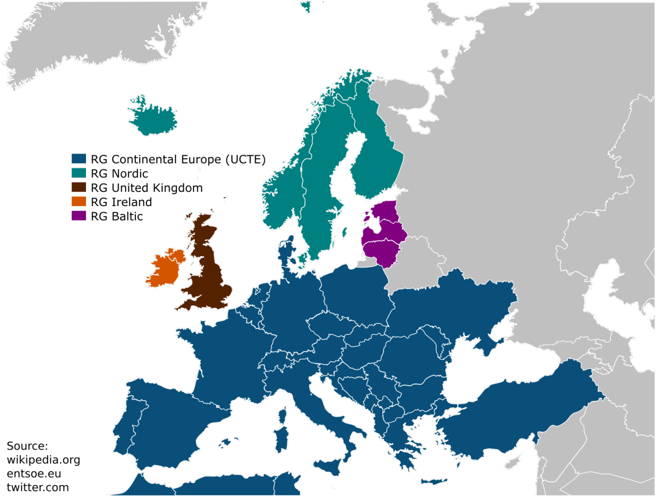
J. Machowski, J. W. Bialek, and J. R. Bumby, *Power System Dynamics*, 2nd ed. (Wiley, Chichester, U.K, 2008).

Synchronous power grid

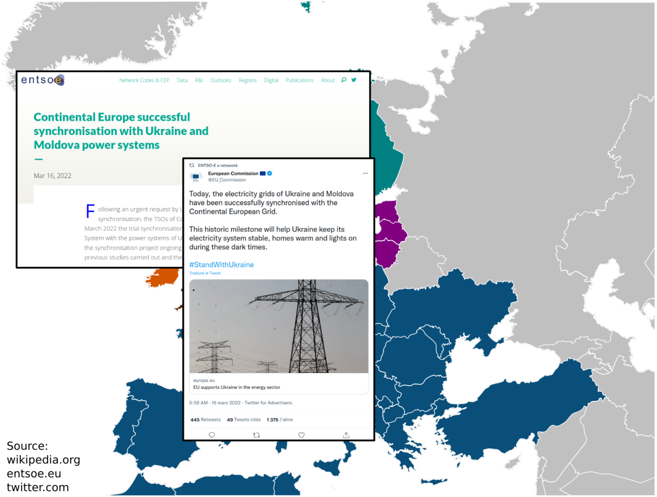
Source: nerc.com



Synchronous power grid

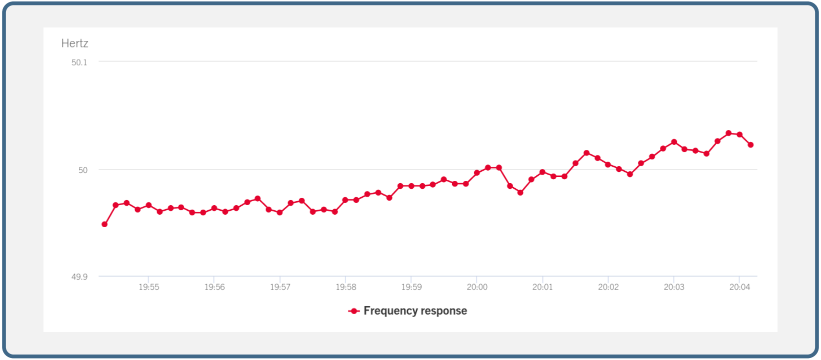


Synchronous power grid



Source:
wikipedia.org
entsoe.eu
twitter.com

Grid frequency



Source: www.swissgrid.ch (Apr. 18, 2022)

The Kuramoto-Sakaguchi model

$$m_j\ddot{\theta}_j + d_j\dot{\theta}_j = P_j - \sum_k B_{jk}\sin(\theta_j - \theta_k) + G_{jk}\cos(\theta_j - \theta_k)$$

$$\dot{\theta}_j = P_j - \sum_k a_{jk}\sin(\theta_j - \theta_k - \phi)$$

$$\dot{\theta}_j = P_j - \sum_k a_{jk}\sin(\theta_j - \theta_k)\,.$$

Assumptions:

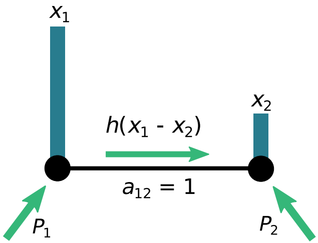
$$m_j = 0\,,\quad d_j = 1\,,\quad a_{jk} = \sqrt{B_{jk}^2 + G_{jk}^2}\,,\quad \phi = \arctan(G_{jk}/B_{jk})\,.$$

H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, *Local and Global Self-Entrainments in Oscillator Lattices*, Prog. Theor. Phys., **77**(5):1005–1010 (1987). doi: 10.1143/PTP.77.1005

Diffusive network - summary

$$\dot{x}_i = P_i - \sum_j a_{ij} h(x_i - x_j)$$

Potentially: $a_{ij} \neq a_{ji}$,
$h(x) \neq -h(-x)$.



- P_i: Natural frequency, commodity injection,...
- a_{ij}: Element of the adjacency matrix;
- h: Coupling function, flow function,...
- x_i: Agent's state.

Main properties

- **Flow conservation**
- Average velocity
- Vectorial form
- Symmetric Jacobian

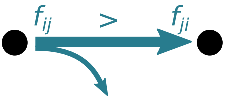
Flow conservation

$$\dot{x}_i = P_i - \sum_j a_{ij} h(x_i - x_j)$$

Undirected coupling: $a_{ij} h(x) = -a_{ji} h(-x)$

[e.g., Kuramoto: $h(x) = \sin(x)$].

Directed coupling: $a_{ij} h(x) \neq -a_{ji} h(-x)$



[Kuramoto-Sakaguchi: $h(x) = \sin(x - \phi)$]

Main properties

- Flow conservation
  $\longrightarrow$ *No flow conservation anymore.*
- **Average velocity**
- Vectorial form
- Symmetric Jacobian

Average velocity

$$\dot{x}_i = P_i - \sum_j a_{ij} h(x_i - x_j)$$

Undirected coupling: $a_{ij} h(x) = -a_{ji} h(-x)$

$$\frac{1}{n} \sum_i \dot{x}_i = \frac{1}{n} \sum_i P_i = \overline{P}$$

Directed coupling: State-dependent $\implies$ time-dependent.

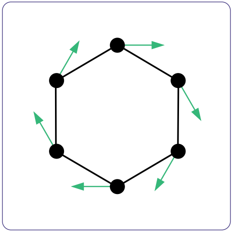
Example: 6-cycle

Cycle of 6 identical Kuramoto-Sakaguchi oscillators.

$$\dot{x}_i = -\sum_j a_{ij} \sin(x_i - x_j - \phi)$$
$$= -\sin(x_i - x_{i-1} - \phi) - \sin(x_i - x_{i+1} - \phi)$$

$$x_0 = (0, 0, 0, 0, 0, 0)$$
$$\implies \dot{x}_i = 2 \sin \phi$$

$$x_1 = (0, \pi/3, 2\pi/3, \pi, -2\pi/3, -\pi/3)$$
$$\implies \dot{x}_i = \sin \phi$$



Main properties

- Flow conservation
  $\longrightarrow$ *No flow conservation anymore.*
- Average velocity
  $\longrightarrow$ *Average velocity is state-/time-dependent.*
- **Vectorial form**
- Symmetric Jacobian

Vectorial form (lossless case)

Linear diffusion:

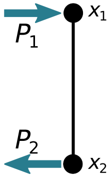
$$\dot{x}_i = P_i - \sum_j a_{ij}(x_i - x_j)\,, \qquad \rightsquigarrow \qquad \dot{x} = P - Lx$$
$$= P - BB^\top x\,.$$

Nonlinear lossless diffusion $[h(x) = -h(-x)]$:

$$\dot{x}_i = P_i - \sum_j a_{ij} h(x_i - x_j)\,, \qquad \rightsquigarrow \qquad \dot{x} = P - Bh(B^\top x)\,.$$

$$h(B^\top x) = \begin{pmatrix} \vdots \\ h(x_i - x_j) \\ \vdots \end{pmatrix} \qquad \begin{aligned} \dot{x}_i &= \cdots - h(x_i - x_j)\cdots \\ \dot{x}_j &= \cdots + h(x_i - x_j)\cdots \end{aligned}$$

Example: 2 agents



$$\dot{x}_1 = P_1 - \sin(x_1 - x_2 - \phi)$$
$$\dot{x}_2 = P_2 - \sin(x_2 - x_1 - \phi)$$

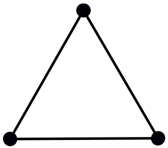
Reminder: $\dot{x} = P - Bh(B^\top x)$.

Let us try: $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, $B = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\dot{x} = P - B\sin(B^\top x - \phi 1_m) = \begin{pmatrix} P_1 - \sin(x_1 - x_2 - \phi) \\ P_2 - \sin(x_2 - x_1 + \phi) \end{pmatrix}$$

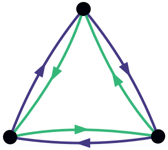
Directed incidence matrices

$$B = \begin{pmatrix} +1 & 0 & +1 \\ -1 & +1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$



$$B_d = (B, -B) = \begin{pmatrix} +1 & 0 & +1 & -1 & 0 & -1 \\ -1 & +1 & 0 & +1 & -1 & 0 \\ 0 & -1 & -1 & 0 & +1 & +1 \end{pmatrix}$$

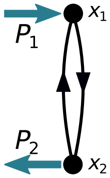
$$B_o = [B_d]_+ = \begin{pmatrix} +1 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & +1 \end{pmatrix}$$



$$L = BB^\top = B_o B_d^\top$$

C. Balbuena, D. Ferrero, X. Marcote, and I. Pelayo, "Algebraic properties of a digraph and its line digraph," J. Interconnect. Netw. 4, 377–393 (2003). doi: 10.1142/S0219265903000933

Example: 2 agents, cont.'d



$$\dot{x}_1 = P_1 - \sin(x_1 - x_2 - \phi)$$
$$\dot{x}_2 = P_2 - \sin(x_2 - x_1 - \phi)$$

Let us try: $B_d = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$, $B_o = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$

$$\dot{x} = P - B_o \left[\sin(B_d^\top x - \phi 1_{2m})\right] = \begin{pmatrix} P_1 - \sin(x_1 - x_2 - \phi) \\ P_2 - \sin(x_2 - x_1 - \phi) \end{pmatrix}$$

$$\dot{x} = P - B_o h(B_d^\top x)$$

Main properties

- Flow conservation
  $\longrightarrow$ *No flow conservation anymore.*
- Average velocity
  $\longrightarrow$ *Average velocity is state-/time-dependent.*
- Vectorial form
  $\longrightarrow$ *There* **exists** *a natural vectorial form.*
- **Symmetric Jacobian**

Symmetric Jacobian

(i) $h(x) = -h(-x) \implies \frac{\partial}{\partial x_j} h(x_i - x_j) = \frac{\partial}{\partial x_i} h(x_j - x_i).$

(ii) $a_{ij} = a_{ji}.$

$$\text{(i) \& (ii)} \implies [J(x)]_{ij} = [J(x)]_{ji}.$$

1. $\lambda_1 = 0$;
2. Eigenvalues are real;
3. Eigenvectors are orthonormal.

For directed coupling:

1. Preserved;
2. No guarantee;
3. Not true in general.

Main properties

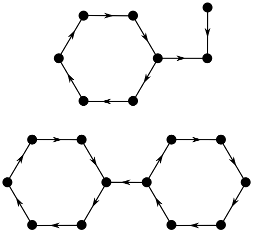
- Flow conservation
  $\longrightarrow$ *No flow conservation anymore.*
- Average velocity
  $\longrightarrow$ *Average velocity is state-/time-dependent.*
- Vectorial form
  $\longrightarrow$ *There* ***exists*** *a natural vectorial form.*
- Symmetric Jacobian
  $\longrightarrow$ *Linearization is not symmetric.*

Synchronization of oscillator systems

Synchronization in the oriented Kuramoto model

- "Kuramoto model": $\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$.
- "Oriented": $a_{ij} \neq 0 \implies a_{ji} = 0$.
- "Synchronization": $\dot{\theta}_i = \dot{\theta}_j$

- Acyclic;
- Homogeneous cycle;
- Combination of the above.



RD, P. Jacquod, and F. Dörfler, *The Kuramoto Model on Oriented and Signed Graphs*, SIAM J. Appl. Dyn. Syst. **18**(1):458-480 (2019). doi: 10.1137/18M1203055

Dynamics on the n-torus

Kuramoto: 2π-periodic coupling.

$$\dot{x}_i = P_i - \sum_j a_{ij} \sin(x_i - x_j)$$

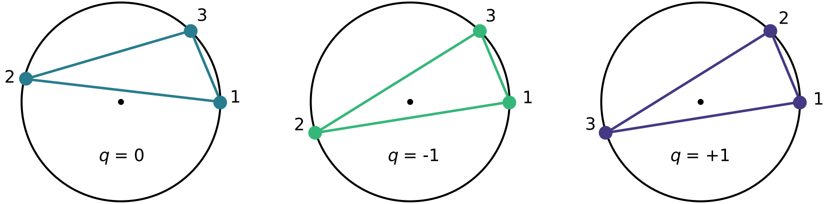
From Euclidean space to the torus:

$$x_i \in \mathbb{R} \quad \to \quad \theta_i \in \mathbb{S}^1 = [-\pi, \pi)$$
$$x \in \mathbb{R}^n \quad \to \quad \theta \in (\mathbb{S}^1)^n = \mathbb{T}^n$$

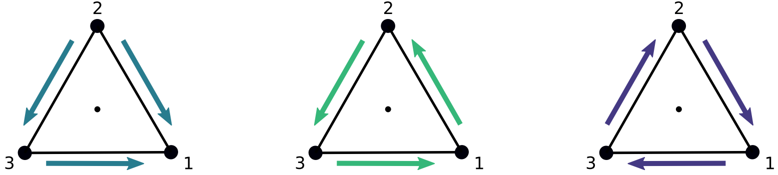
$\to$

Winding number and loop flows

Given a cycle $\sigma = (i_1, ..., i_\ell)$: $\quad q = (2\pi)^{-1} \sum_{k=1}^{\ell} d_{cc}(\theta_{i_k}, \theta_{i_{k-1}}) \in \mathbb{Z}.$



Loop flows:



Winding vectors and partition

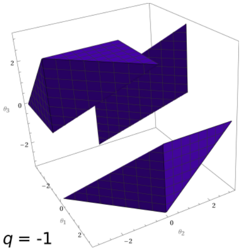
For a cycle σ.
The **winding number**:

$$q_\sigma : \mathbb{T}^n \to \mathbb{Z}$$
$$\theta \mapsto q_\sigma(\theta)$$

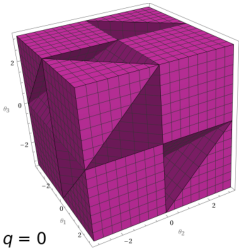
For a cycle basis $\Sigma = (\sigma_1, \dots, \sigma_c)$.
The **winding vector**:

$$q_\Sigma : \mathbb{T}^n \to \mathbb{Z}^c$$
$$\theta \mapsto [q_{\sigma_1}(\theta), \dots, q_{\sigma_c}(\theta)]$$

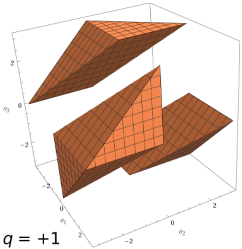
Winding cells: $\Omega_u = \{\theta \mid q_\Sigma(\theta) = u\}$.



q = -1



q = 0



q = +1

Synchronization in Kuramoto-Sakaguchi

$$\dot{x} = P - B_o h(B_d^\top x) = P - B_o \left[h_c(B_d^\top x) + h_d(B_d^\top x) \right]$$

Theorem

Consider a connected network of coupled oscillators. For sufficiently low dissipation to coupling ratio,

$$d_{\max} \cdot L_d < \lambda_2 \cdot L_c \,,$$

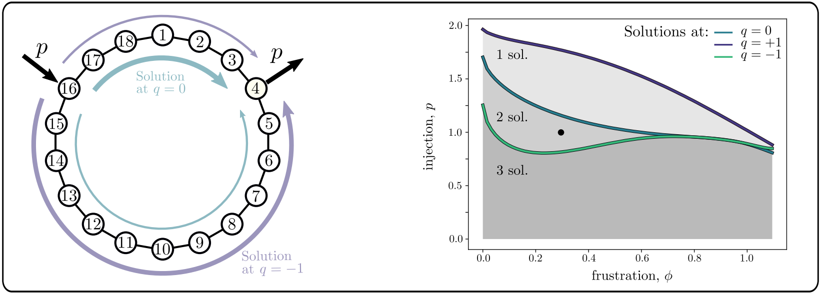
there is **at most one** *stable synchronous state in each winding cell.*

RD, S. Jafarpour, and F. Bullo, *Multistability and Paradoxes in Lossy Oscillator Networks*, arXiv preprint: 2202.02439 (2022).

S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo, *Flow and Elastic Networks on the n-Torus: Geometry, Analysis, and Computation*, SIAM Review **64** (2022). doi: 10.1137/18M1242056

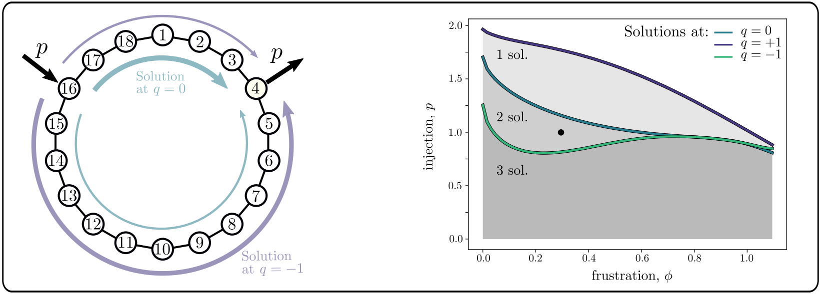
Paradox 1: "Loop flows increase capacity."

$$\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j - \phi)$$



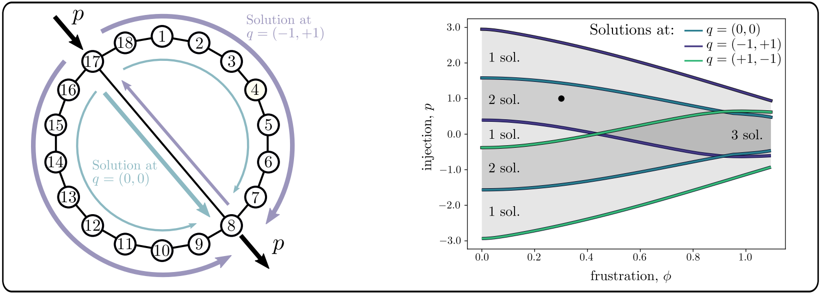
Paradox 2: "Frustration increases capacity."

$$\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j - \phi)$$



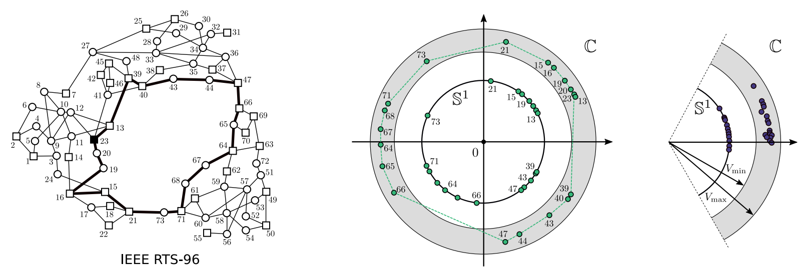
Paradox 3: "Frustration promotes multistability."

$$\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j - \phi)$$



Summary:
- Properties preserved and not;
- Synchronization of networked oscillators;
- Paradoxes in lossy oscillator networks.

Follow-ups:
- Relax the assumptions for *at most uniqueness*;
- Refine the power flow models.



Thank you!

Balbuena et al.

Consider a weighted directed graph and let:

- A: adjacency matrix;
- D_o, D_i: out- and in-degree matrices;
- $L = D_o - A$: Laplacian matrix;
- B: incidence matrix.
- $B_o = [B]_+$, $B_i = [B]_-$: out- and in-incidence matrices;
- W: edge weight matrix.

Then,

$$D_o = B_o W B_o^\top , \quad D_i = B_i W B_i^\top , \quad A = B_o W B_i^\top , \quad L = B_o W B^\top .$$

C. Balbuena, D. Ferrero, X. Marcote, and I. Pelayo, "Algebraic properties of a digraph and its line digraph," J. Interconnect. Netw. 4, 377–393 (2003). doi: 10.1142/S0219265903000933

Derivation of the power flow equations

Define:

- Current: $I_j \in \mathbb{C}$;
- Voltage: $V_j = |V_j|e^{i\theta_j}$;
- Power: $S_j = P_j + iQ_j$;
- Impedance: $Z_{jk} = R_{jk} + iX_{jk}$;
- Admittance: $Y_{jk} = Z_{jk}^{-1} = G_{jk} + iB_{jk}$.

Compute the power:

$$P_j + iQ_j = V_j I_j^* = V_j \left[\sum_k Z_{jk}^{-1}(V_k - V_j) \right]^* = \sum_k Y_{jk}^* |V_j||V_k| e^{i(\theta_j - \theta_k)}$$

$$= \sum_k \left(V_j V_k \left[B_{jk} \sin(\theta_j - \theta_k) + G_{jk} \cos(\theta_j - \theta_k) \right] \right.$$

$$\left. + \, i V_j V_k \left[G_{jk} \sin(\theta_j - \theta_k) - B_{jk} \cos(\theta_j - \theta_k) \right] \right) .$$

Idea of the proof

1. Define the iteration map leaving the winding cells invariant:

$$\begin{aligned} S_\epsilon \colon \mathbb{R}^m &\to \mathbb{R}^m \\ \mathbf{\Delta} &\mapsto \mathbf{\Delta} - \epsilon B^\top L^\dagger \left[B_o h(\mathbf{\Delta}) - P\right] . \end{aligned}$$

2. Decompose the coupling function:

$$h_f(x) = \left[h(x) - h(-x)\right]/2\,, \qquad h_l(x) = \left[h(x) + h(-x)\right]/2$$

3. Construct the state dependent graphs with Laplacian matrices

$$\begin{aligned} L_f(x) &= B \cdot \mathrm{diag}(\cdots, h_f'(x_i - x_j), \cdots) \cdot B^\top\,, \\ L_l(x) &= B \cdot \mathrm{diag}(\cdots, |h_l'(x_i - x_j)|, \cdots) \cdot B^\top\,. \end{aligned}$$

4. Compare $\lambda_2(L_f)$ and $\mathrm{diag}(L_l)$.