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Reconstructing Network Structure from Partial Measurements

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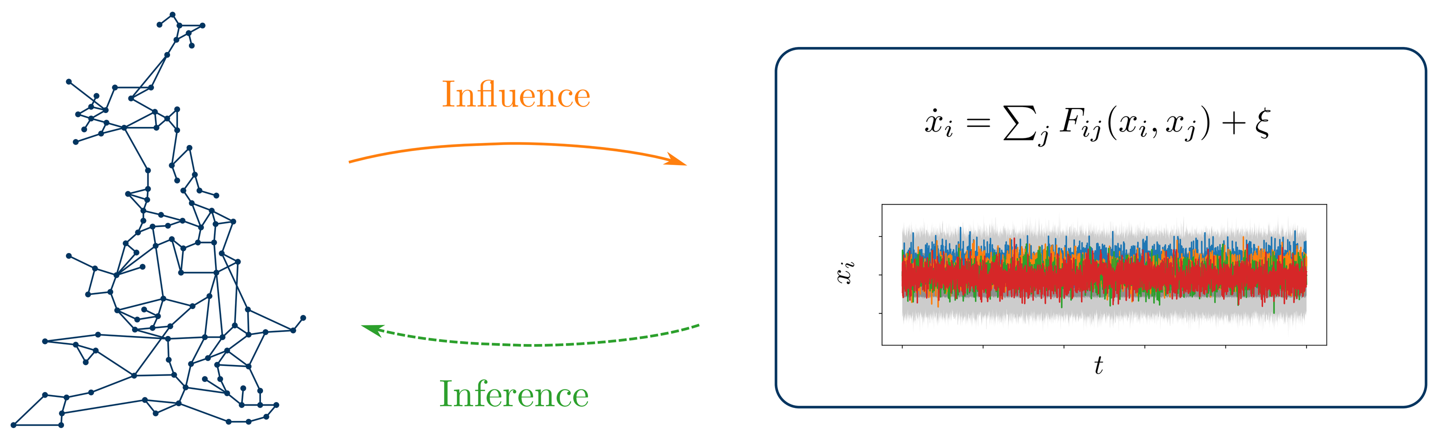
Interplay between Complex Networks and Dynamical Systems



Claim: The information about the network structure can be recovered from observation of the agents' dynamics.

Framework and Linearized Dynamics

We consider dynamical systems modeled by the autonomous ordinary differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}[\boldsymbol{x}(t)] + \boldsymbol{\xi}(t)\,, \tag{1}$$

in the vicinity of a **stable fixed point** $\boldsymbol{x}^*$. We further assume that the Jacobian matrix

$$\mathbb{J}_{ij}(\boldsymbol{x}^*) = -\frac{\partial F_i(\boldsymbol{x}^*)}{\partial x_j}\,,$$

is **real symmetric negative semidefinite**.

In particular, this cover the case of **networks of diffusively coupled dynamical agents**, e.g., high-voltage power grids, some social networks, and chemical reactions.

Close to $\boldsymbol{x}^*$, the time evolution of the deviation $\boldsymbol{\delta x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*$ reasonably approximated by the linearized dynamics

$$\dot{\boldsymbol{\delta x}} \approx -\mathbb{J}(\boldsymbol{x}^*)\boldsymbol{\delta x} + \boldsymbol{\xi}\,, \tag{2}$$

where $\boldsymbol{\xi}$ represents the unavoidable **environmental noise**.

In this work, we assume that the noise's first two moments are homogeneous throughout the network, i.e.,

$$\langle \xi_i(t)\rangle = 0\,, \qquad \langle \xi_i(t_1)\xi_j(t_2)\rangle = \xi_0^2\delta_{ij}\exp\left(-|t_1-t_2|/\tau_0\right)\,.$$

Let us denote by
- $\lambda_\alpha \in \mathbb{R}$: The αth eigenvalue of $\mathbb{J}$, for $\alpha\in\{1,...,n\}$, ordered such that $0\le\lambda_1\le...\le\lambda_n$;
- $\boldsymbol{u}_\alpha\in\mathbb{R}^n$: The corresponding eigenvector of $\mathbb{J}$, which, form a orthonormal basis of $\mathbb{R}^n$.

Decomposing the deviation $\boldsymbol{\delta x}$ on the eigenbasis of $\mathbb{J}$ yields,

$$\boldsymbol{\delta x}(t) = \sum_{\alpha=1}^n c_\alpha(t)\boldsymbol{u}_\alpha\,, \qquad \dot c_\alpha = -\lambda_\alpha c_\alpha + \boldsymbol{u}_\alpha^\top\boldsymbol{\xi}\,,$$

which has solution

$$c_\alpha(t) = e^{-\lambda_\alpha t}\int_0^t e^{\lambda_\alpha t'}\boldsymbol{u}_\alpha^\top\boldsymbol{\xi}(t')\mathrm{d}t'\,.$$

Direct Network Reconstruction

Let us now expand the velocities $\dot{\boldsymbol{\delta x}}$ on the eigenmodes of $\mathbb{J}$ and compute their **two-point correlators**

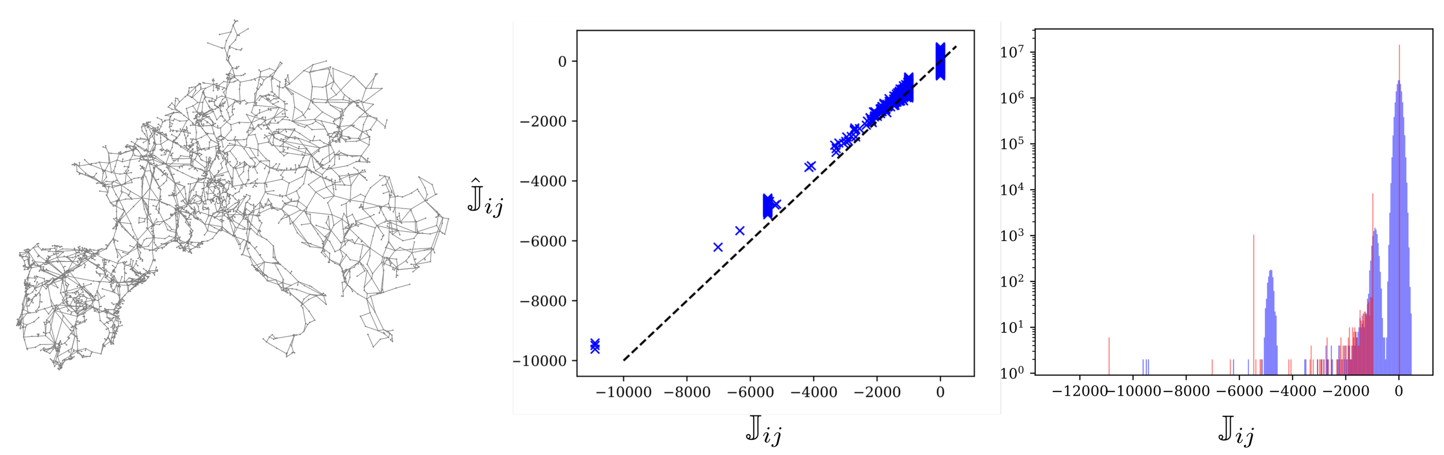
$$\lim_{t\to\infty}\langle\dot{\boldsymbol{\delta x}}_i(t)\dot{\boldsymbol{\delta x}}_j(t)\rangle = \lim_{t\to\infty}\sum_{\alpha,\beta}\langle\dot c_\alpha(t)\dot c_\beta(t)\rangle u_{\alpha,i}u_{\beta,j} = \xi_0^2\left(\delta_{ij} - \sum_{\alpha\ge1}u_{\alpha,i}u_{\alpha,j}\frac{\lambda_\alpha\tau_0}{1+\lambda_\alpha\tau_0}\right)\,. \tag{3}$$

A series expansion of Eq. (3) in the limit $\lambda_\alpha\tau_0 < 1$ gives

$$\lim_{t\to\infty}\langle\dot{\boldsymbol{\delta x}}_i(t)\dot{\boldsymbol{\delta x}}_j(t)\rangle = \xi_0^2\left[\delta_{ij} + \sum_{k=1}^\infty(-\tau_0)^k(\mathbb{J}^k)_{ij}\right]\,. \tag{4}$$

For sufficiently small correlation time of the noise, $\lambda_\alpha\tau_0\to0$, the two-point correlators are dominated by the $k=1$ term, which allows to approximate

$$\hat{\mathbb{J}}_{ij} = \frac{\delta_{ij} - \langle\dot{\delta x}_i\dot{\delta x}_j\rangle/\xi_0^2}{\tau_0}\,.$$

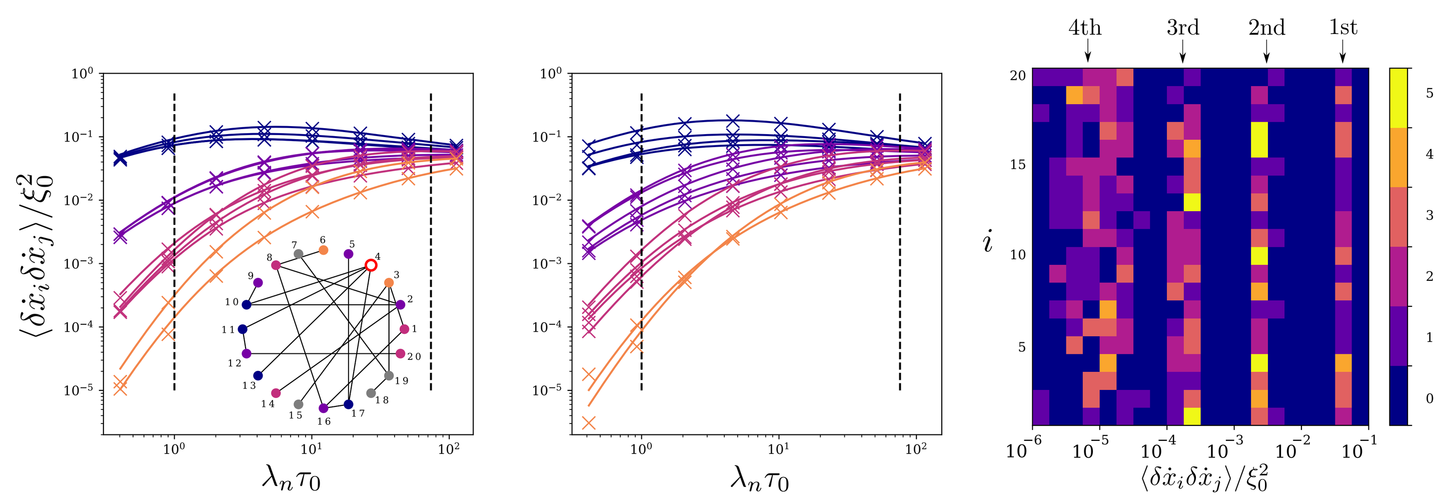


Inferring Geodesic Distances

One notices that if nodes i and j are a geodesic distance $q>1$, all terms with $k<q$ in Eq. (4) vanish, which gives

$$\lim_{t\to\infty}\langle\dot{\boldsymbol{\delta x}}_i(t)\dot{\boldsymbol{\delta x}}_j(t)\rangle = \xi_0^2\left[\delta_{ij} + \sum_{k=q}^\infty(-\tau_0)^k(\mathbb{J}^k)_{ij}\right]\,. \tag{5}$$

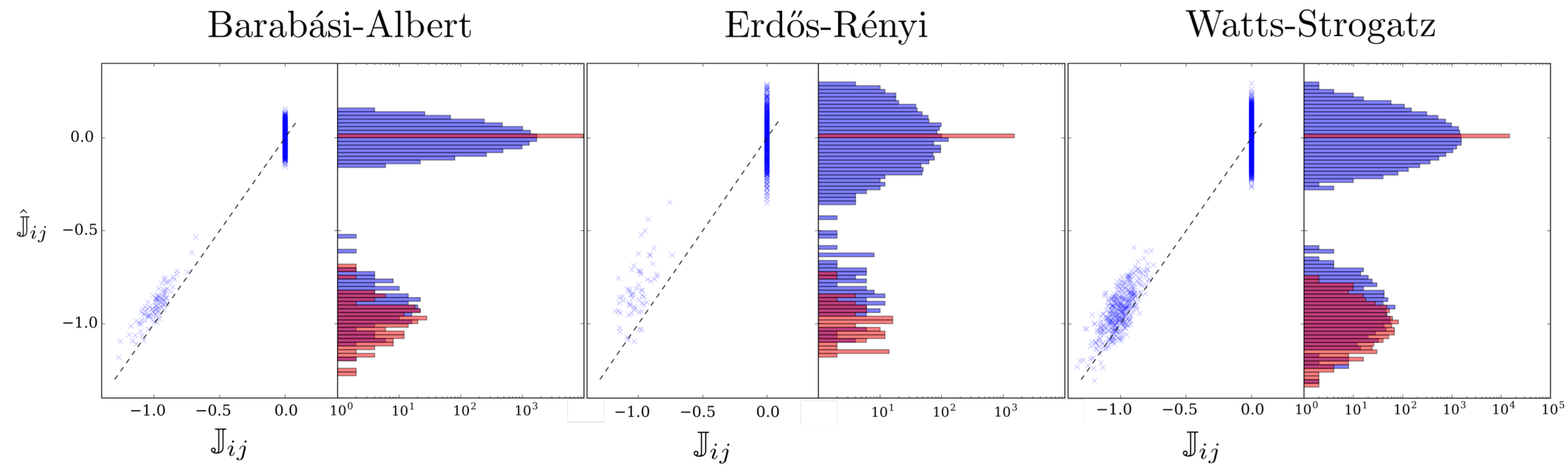
Again, for small noise correlation time, the dominant term in Eq. (5) is $k=q$. Under mild conditions of homogeneity of the couplings, this allows to determine the geodesic distance up to some accuracy.



Partial Reconstruction from Partial Measurements

Even if one has access only to measurements at a subset of agents, it is still possible to precisely identify the connectivity between observed agents.

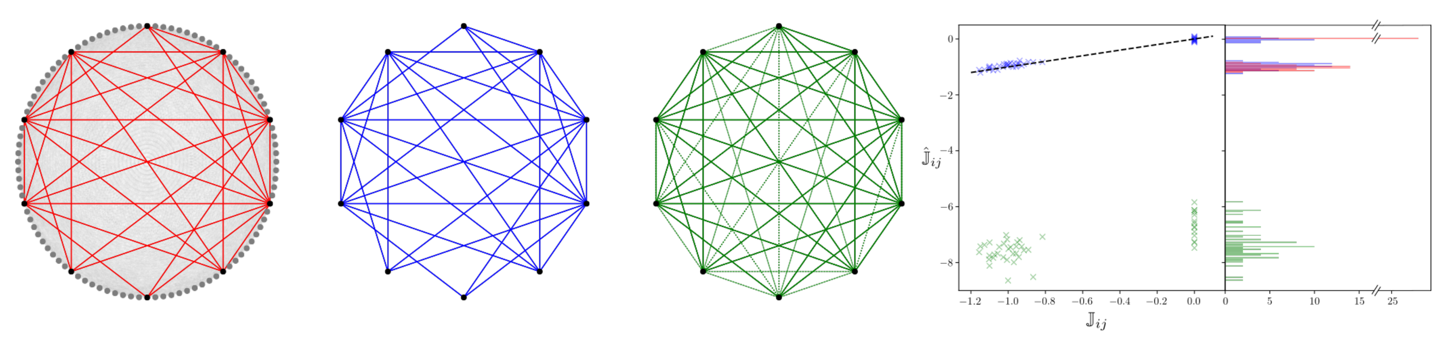
We show the result of our method on three random networks with 1000 nodes from which only 100 are observed [2,3].



Related Approaches

Very similar approaches where proposed in the literature, e.g., Ref. [4]. However, as far as we can tell, all of them suffer from the necessity of a matrix inverstion. Namely, these methods infer the components of the **(pseudo-)inverse of the Jacobian**.

Blindly applying this method compares poorly with ours.



Red and grey: Erdős-Rényi graph with $n=100$ nodes, among which 10 are observed (black dots connected by red lines).
Blue: Edges inferred by our method.
Green: Edges inferred by the method of Ref. [4]. Plain lines are true positives and dashed lines are false positive.

References

[1] M. Tyloo, R. Delabays, and P. Jacquod, Reconstructing Network Structure from Partial Measurements, *arXiv preprint: 2007.16136* (2020).
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