Bounding the destabilization time in networks of coupled noisy oscillators

Robin Delabays

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M. Tyloo, R. D., and Ph. Jacquod, *arXiv preprint* **1812.09497** (2018)

Hes·so VALAIS WALLIS
School of Engineering

Where is HES-SO?

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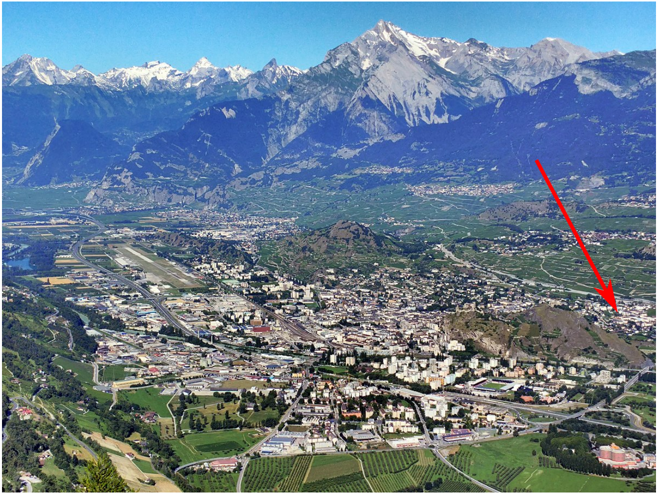


Noisy power injection

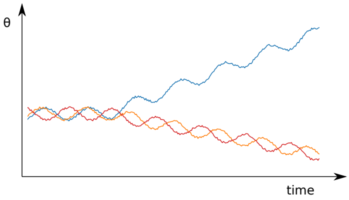
Energy Transition

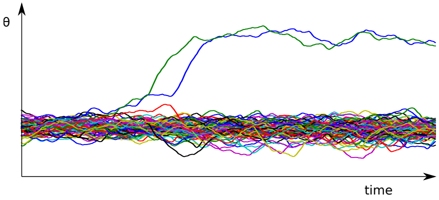
↓

Fluctuating power injections

↓

Possible loss of synchrony or
convergence to a new synchronous state





Questions

How likely is it for a system to lose synchrony?

Can we quantify this probability?

What are the relevant parameter of the fluctuations?

The model – the oscillators

Lossless line approximation of the Swing Equations.

The 2nd order Kuramoto model:

$$m \cdot \ddot{\theta}_i + d \cdot \dot{\theta}_i = P_i(t) - \sum_{j=1}^{n} b_{ij} \sin(\theta_i - \theta_j)\,.$$

- $\theta_i \in \mathbb{S}^1$: voltage angle / oscillator's phase,
- m, d: inertia and damping,
- $P_i \in \mathbb{R}$: power injection / natural frequency,
- b_{ij}: weighted adjacency matrix.

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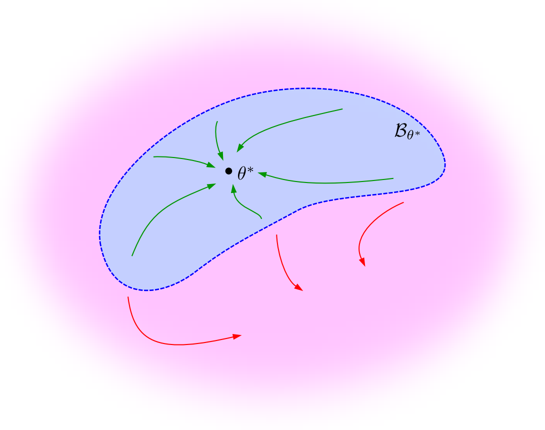
Synchrony?

Basins of attraction

$$\mathcal{B}_{\theta^*} \coloneqq \{\theta^\circ \in \mathbb{T}^n \ : \ \theta(0) = \theta^\circ \implies \theta(t \to \infty) = \theta^*\} \ .$$



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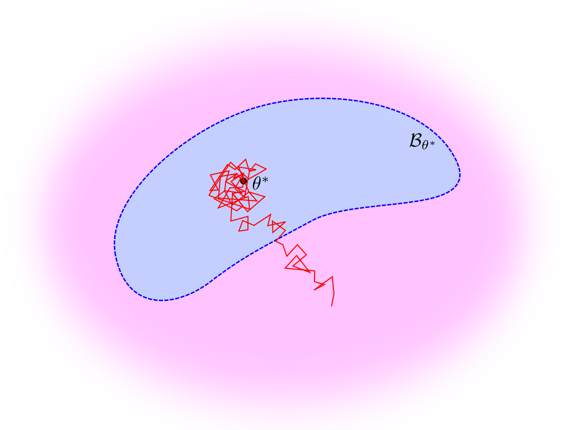
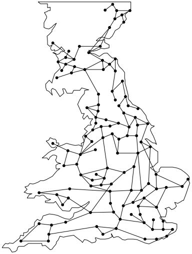


The model – the network



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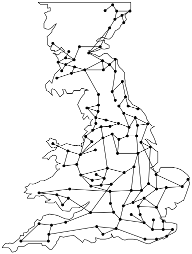

Consider a synchronous state $\boldsymbol{\theta}^{(0)} \in \mathbb{T}^n$
We define the weighted Laplacian (Jacobian),

$$\mathbb{L}_{ij} := \begin{cases} -b_{ij} \cos\left(\theta_i^{(0)} - \theta_j^{(0)}\right), & i \neq j, \\ \sum_{k \neq i} b_{ik} \cos\left(\theta_i^{(0)} - \theta_k^{(0)}\right), & i = j. \end{cases}$$

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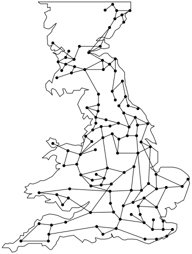

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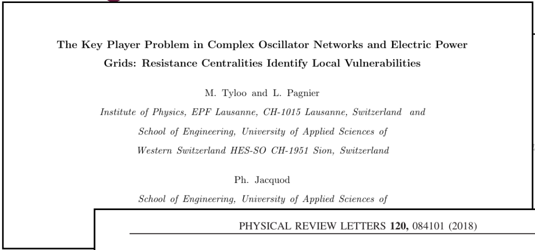
Eigen decomposition:

$$\lambda_1 = 0 < \lambda_2 \leq ... \leq \lambda_n$$

$$\mathbf{u}_1 \sim (1, ..., 1) , \qquad \mathbf{u}_\alpha \perp \mathbf{u}_1 , \ \alpha \geq 2$$

Advertising

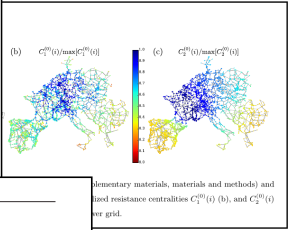

The Key Player Problem in Complex Oscillator Networks and Electric Power Grids: Resistance Centralities Identify Local Vulnerabilities

M. Tyloo and L. Pagnier

Institute of Physics, EPF Lausanne, CH-1015 Lausanne, Switzerland and School of Engineering, University of Applied Sciences of Western Switzerland HES-SO CH-1951 Sion, Switzerland

Ph. Jacquod

School of Engineering, University of Applied Sciences of



plementary materials, materials and methods) and
lized resistance centralities $C_1^{(0)}(i)$ (b), and $C_2^{(0)}(i)$
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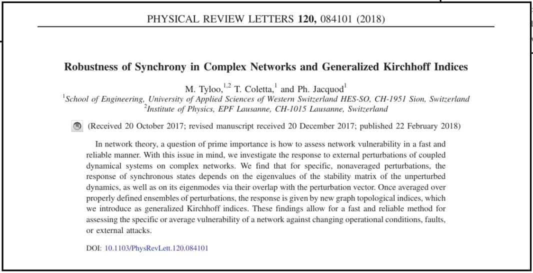
PHYSICAL REVIEW LETTERS **120,** 084101 (2018)

Robustness of Synchrony in Complex Networks and Generalized Kirchhoff Indices

M. Tyloo,[1,2] T. Coletta,[1] and Ph. Jacquod[1]
[1]*School of Engineering, University of Applied Sciences of Western Switzerland HES-SO, CH-1951 Sion, Switzerland*
[2]*Institute of Physics, EPF Lausanne, CH-1015 Lausanne, Switzerland*

(Received 20 October 2017; revised manuscript received 20 December 2017; published 22 February 2018)

In network theory, a question of prime importance is how to assess network vulnerability in a fast and reliable manner. With this issue in mind, we investigate the response to external perturbations of coupled dynamical systems on complex networks. We find that for specific, nonaveraged perturbations, the response of synchronous states depends on the eigenvalues of the stability matrix of the unperturbed dynamics, as well as on its eigenmodes via their overlap with the perturbation vector. Once averaged over properly defined ensembles of perturbations, the response is given by new graph topological indices, which we introduce as generalized Kirchhoff indices. These findings allow for a fast and reliable method for assessing the specific or average vulnerability of a network against changing operational conditions, faults, or external attacks.

DOI: 10.1103/PhysRevLett.120.084101

M. Tyloo, T. Coletta, and Ph. Jacquod, *Phys. Rev. Lett.* **120** (2018).

M. Tyloo, L. Pagnier, and Ph. Jacquod, *arXiv preprint* **1810.09694** (2018).

The model – the noise

We consider noisy power injections,

$$P_i(t) := P_i^{(0)} + \delta P_i(t)\,,$$

such that

$$\left\langle \delta P_i(t) \cdot \delta P_j(t') \right\rangle = \delta_{ij} \cdot \delta P_0^2 \cdot \exp(-|t - t'|/\tau_0)\,,$$

where $\tau_0 > 0$ is the correlation time.

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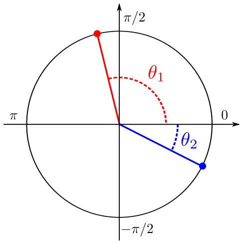
Simulations: We construct a Gaussian noise with correlation time τ_0.

Three time scales

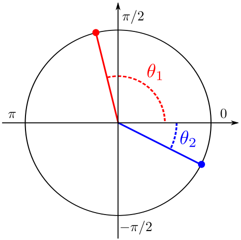
Three time scales

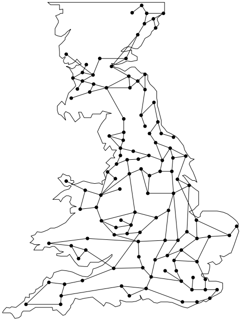
Oscillators: $\dfrac{m}{d}$,



Three time scales

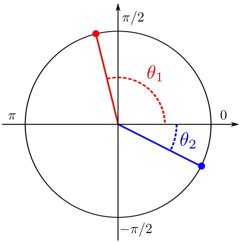
Oscillators: $\dfrac{m}{d}$, **Network:** $\dfrac{d}{\lambda_\alpha}$,

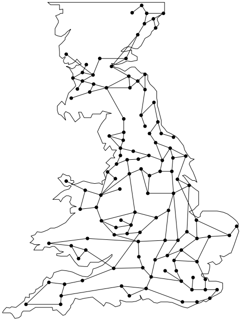




Three time scales

Oscillators: $\dfrac{m}{d}$, **Network:** $\dfrac{d}{\lambda_\alpha}$, **Noise:** τ_0.





Analytical prediction

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and linearize the dynamics

$$m\delta\ddot{\theta}_i + d\delta\dot{\theta}_i = P_i^{(0)} + \delta P_i(t) - \sum_j b_{ij}\sin(\theta_i^{(0)} + \delta\theta_i - \theta_j^{(0)} - \delta\theta_j)\,.$$

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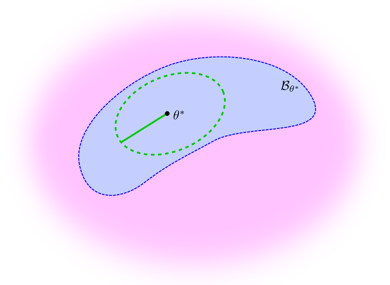
Expanding on the eigenmodes and taking $t \to \infty$,

$$\langle\delta\boldsymbol{\theta}^2\rangle = \left\langle \left(\sum_\alpha c_\alpha \mathbf{u}_\alpha\right)^2 \right\rangle = \delta P_0^2 \sum_{\alpha\geq 2} \frac{\tau_0 + m/d}{\lambda_\alpha\left(\lambda_\alpha\tau_0 + d + m/\tau_0\right)}$$

Analytical prediction

Typical distance from the sync state after a long time:

$$\delta P_0^2 \sum_{\alpha \geq 2} \frac{\tau_0 + m/d}{\lambda_\alpha \left(\lambda_\alpha \tau_0 + d + m/\tau_0\right)} \,.$$



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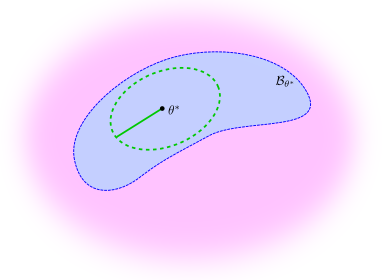

Can we derive a condition for loss of synchrony?

Escape from the basin

Potential:

$$V(\boldsymbol{\theta}) = -\sum_i P_i\theta_i + \sum_{i<j} b_{ij}\left(1 - \cos(\theta_i - \theta_j)\right) .$$

L. DeVille, *Nonlinearity* **25** (2012).

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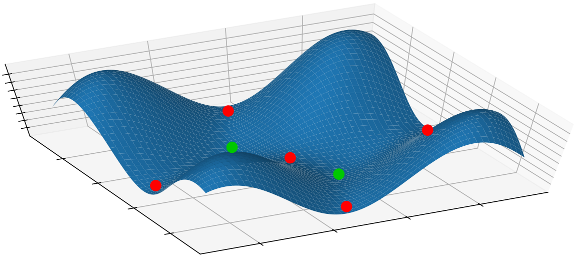
Almost surely, escape occurs through a 1-saddle φ, of the noiseless system.



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Idea

For given network and noise, compare:

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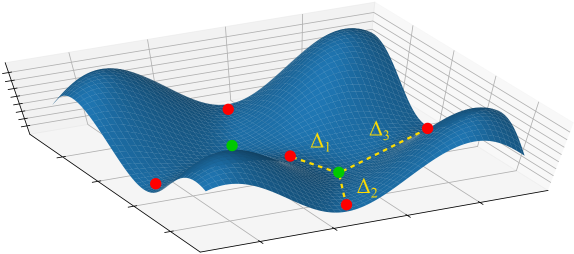
The typical excursion: $\delta P_0^2 \sum_{\alpha \geq 2} \frac{\tau_0 + m/d}{\lambda_\alpha \left(\lambda_\alpha \tau_0 + d + m/\tau_0\right)}$;

Idea

For given network and noise, compare:

The typical excursion: $$\delta P_0^2 \sum_{\alpha \geq 2} \frac{\tau_0 + m/d}{\lambda_\alpha \left(\lambda_\alpha \tau_0 + d + m/\tau_0\right)}\,;$$

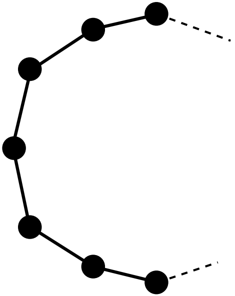
and the distance Δ to the closest 1-saddle.



Cycle network, $n = 83$, $P_i^{(0)} \equiv 0$, $m = 0$

Distance between stable sync state $\boldsymbol{\theta}^{(0)}$ and the closest 1-saddle $\boldsymbol{\varphi}^{(1)}$ is

$$\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}^{(1)}\|_2^2 = \frac{n(n^2-1)}{12(n-2)^2}\pi^2 =: \Delta\,.$$

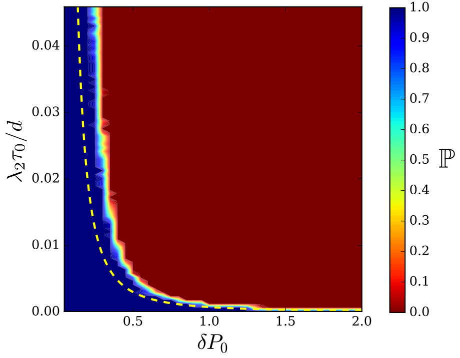


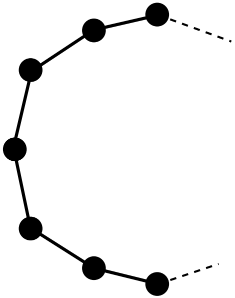
L. DeVille, *Nonlinearity* **25** (2012).

R. D., M. Tyloo, and Ph. Jacquod, *Chaos* **27** (2017).

Cycle network, $n = 83$, $P_i^{(0)} \equiv 0$, $m = 0$

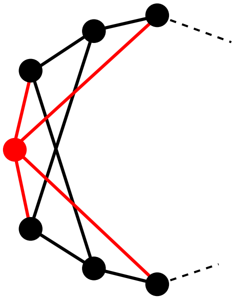
$$\delta P_0^2 \sum_{\alpha \geq 2} \frac{\tau_0}{\lambda_\alpha \left(\lambda_\alpha \tau_0 + d\right)} = \Delta^2 \,.$$



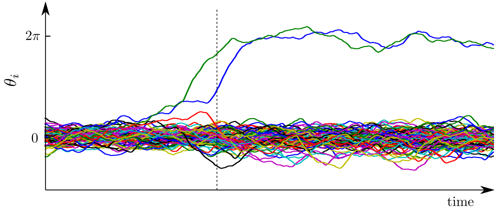


Cycle with 3rd neighbor, $n = 83$, $P_i^{(0)} \equiv 0$, $m = 0$

How can we determine Δ for arbitrary network.



Locating 1-saddles



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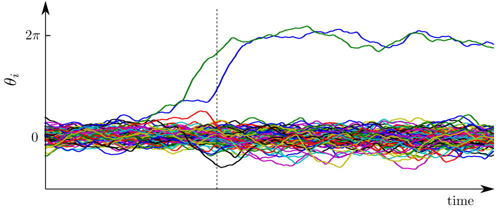
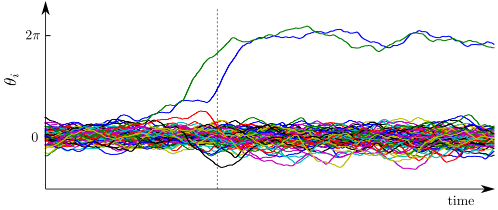


Locate a candidate $\boldsymbol{\theta}^\circ \longrightarrow$ initial conditions.

L. DeVille, *Nonlinearity* **25** (2012).

Locating 1-saddles



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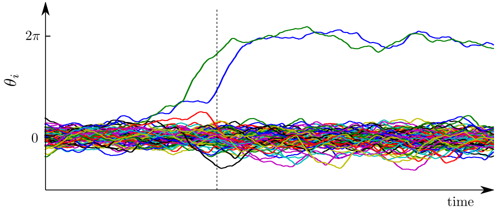

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Check eigenvalues of $\mathbb{L}(\boldsymbol{\theta}^*)$.

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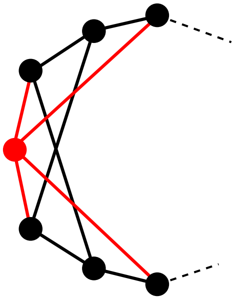
Cycle with 3rd neighbor, $n = 83$, $P_i^{(0)} \equiv 0$, $m = 0$

Unique significant 1-saddle.



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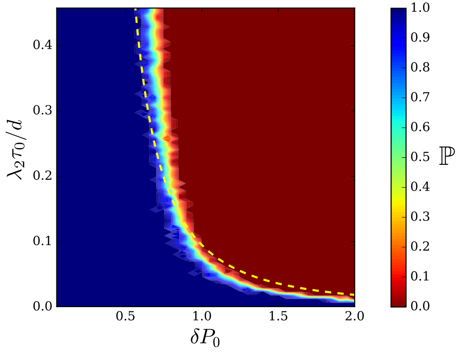
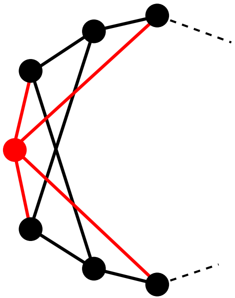
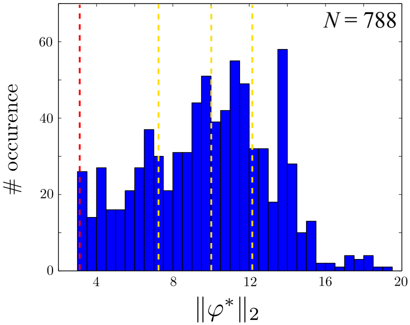
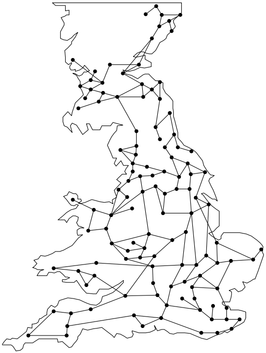




UK network, $n = 120$, $P_i^{(0)} \equiv 0$, $m = 0$





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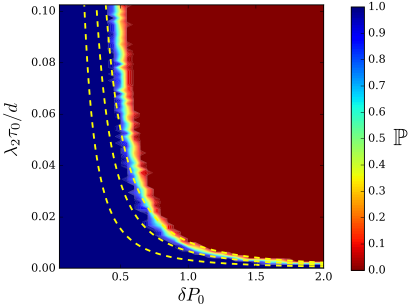
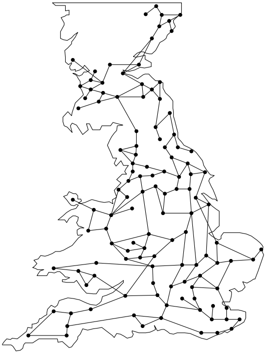
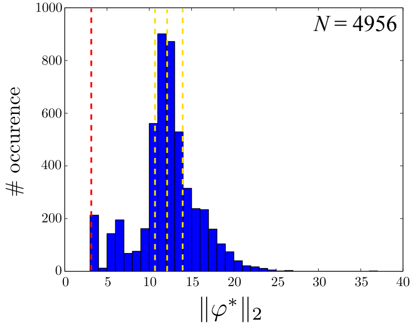
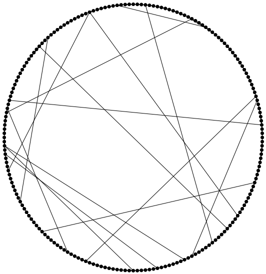




Small world, $n = 200$, $P_i^{(0)} \equiv 0$, $m = 0$





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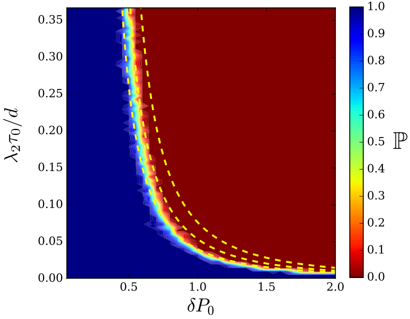
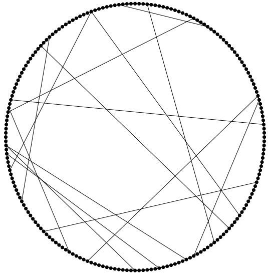
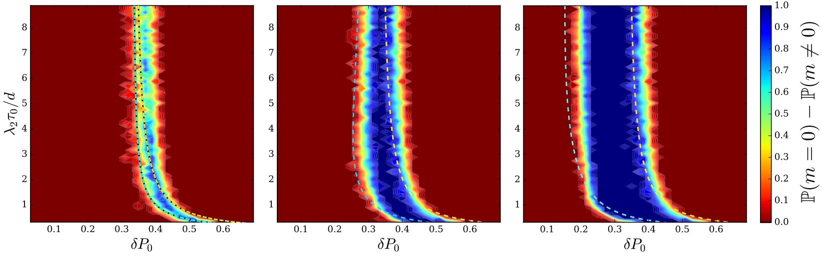




The effect of inertia

Cycle with 3rd neighbor.



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$d/\lambda_2 = 0.35$

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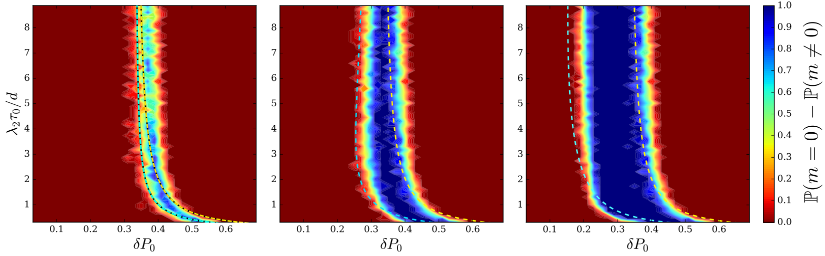

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Inertia seems not to stabilize the system!

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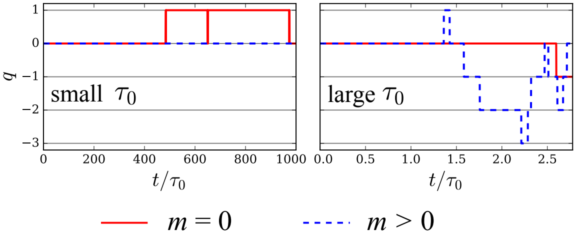
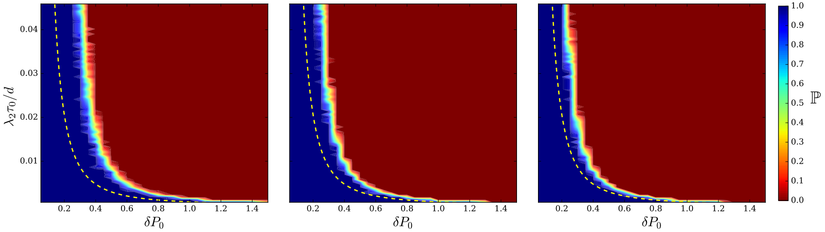


Inertia seems not to stabilize the system!
Not always at least.

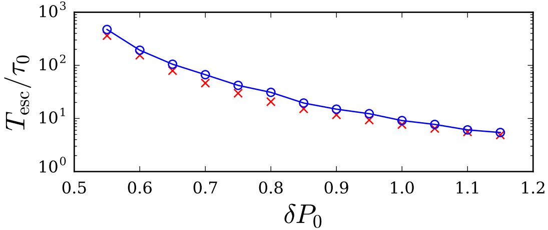
Superexponential escape time

Cycle, $n = 83$.



Superexponential escape time

Cycle, $n = 83$.



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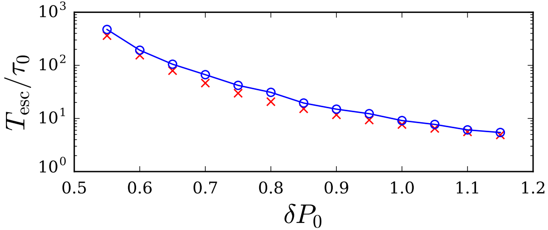

$$T_{\mathrm{esc}} \propto \left[2 \int_{\beta\Delta}^{\infty} P(\overline{\delta\theta}) d(\overline{\delta\theta}) \right]^{-1}$$

Conclusion

- Qualitatively describe the boundary between stable and unstable parameter regions;
- Inertia does not stabilizes the network (in this setting);
- Numerical method to locate 1-saddles.

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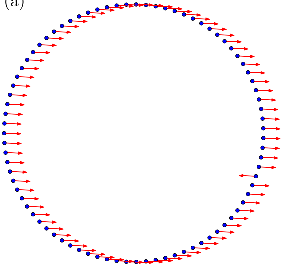
Further work:

- Plug in "real-life" parameters;
- Quantify the precision of our prediction.

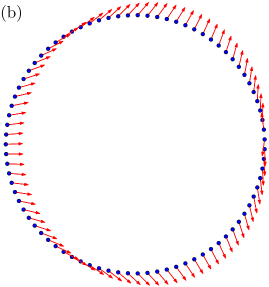
Thank you!

The 1-saddles for the cycle with 3$^{\mathrm{rd}}$ neighbor

(a)

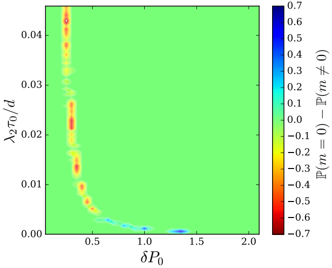


(b)

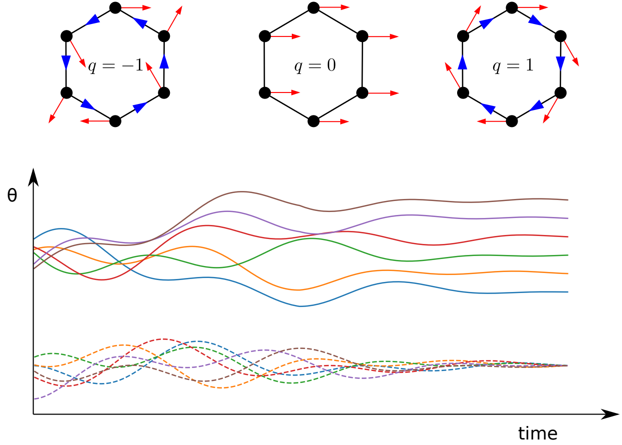


Inertia 2

Cycle, $n = 83$, $m/d = 10$, and $d/\lambda_2 = 175$.



Multistability



R. D., T. Coletta, and Ph. Jacquod, *J. Math. Phys.* **57** (2016)
R. D., T. Coletta, and Ph. Jacquod, *J. Math. Phys.* **58** (2017)