The Size of the Sync Basin Revisited

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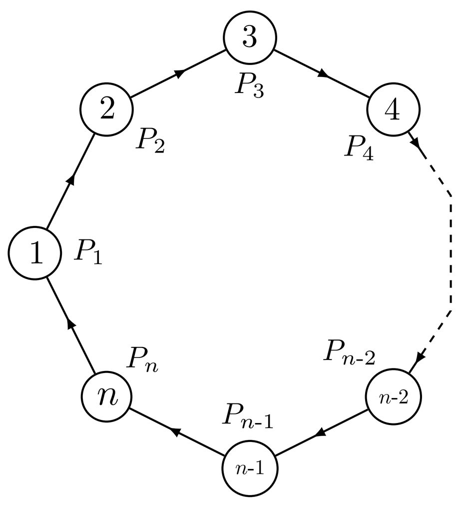
Motivation

It is known since the work of Korsak [1] that stable equilibria of the Swing Equations differ by **loop flows**. Linear stability of these fixed points have been extensively investigated through Lyapunov exponents. We explore the basins of attraction of fixed points of the **Kuramoto model** with cyclic interactions describing the behavior of n nonlinearly coupled oscillators

$$\dot{\theta}_i = P_i - K\sin(\theta_i - \theta_{i-1}) - K\sin(\theta_i - \theta_{i+1})\,, \qquad (1)$$

$i = 1, ..., n$, where

$$\theta_i \in \mathbb{R}\,, \qquad K > 0\,, \qquad \sum_i P_i = 0.$$



Stable fixed points of Eq. (1) are characterized by their integer **winding number**

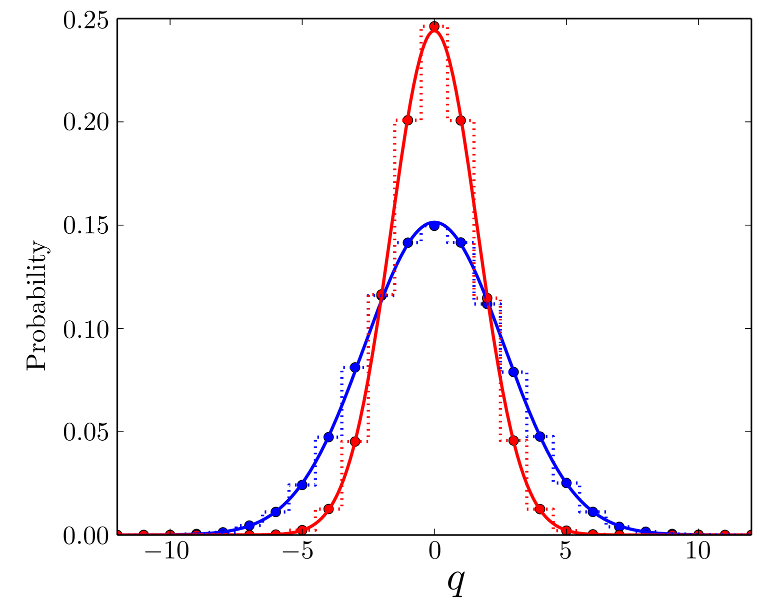
$$q = \frac{1}{2\pi}\sum_{i=1}^{n}\Delta_{i,i+1} \in \mathbb{Z}\,, \qquad \Delta_{i,i+1} := [\theta_i - \theta_{i+1} \pmod{2\pi}]\,. \qquad (2)$$

Wiley et al. [2]: The relation between an equilibrium's winding number and the volume of its basin of attraction is **Gaussian** (red dots).

But: Initial and final winding numbers are **correlated** (correlation coefficient 0.47).

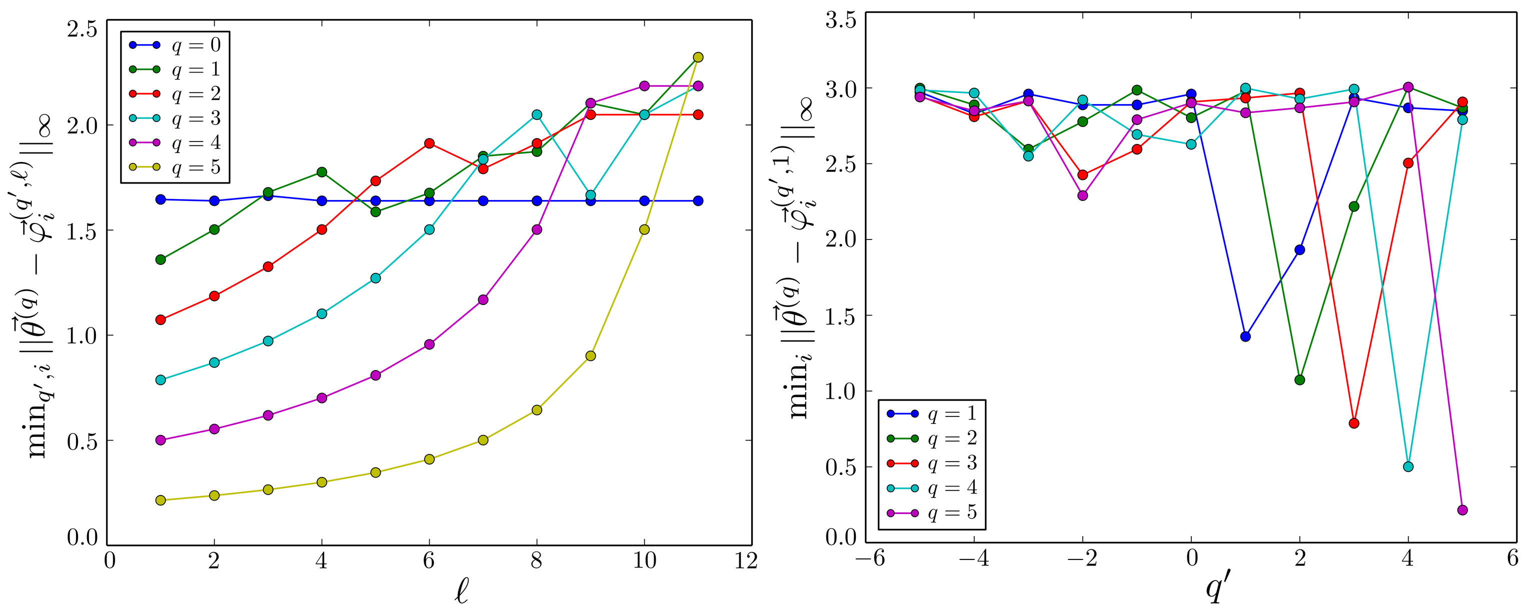
As the state space has **high dimensionality**, a very **large number** of initial conditions is then necessary to cover the whole state space $[(2\pi/0.5)^{80} \approx 10^{88}]$.

We propose a **new method** to assess the basins of attraction's volumes.



Identical frequencies: Analytical approach

In the case of identical frequencies ($P_i \equiv 0$), we observed that a stable fixed point $\vec{\theta}^{(q)}$ and a p-saddle $\vec{\varphi}^{(q')}$ are the closest if $q = q'$ and $p = 1$.



For identical frequencies, a fixed point has to satisfy

$$\Delta_{i,i+1} = \Delta_{i-1,i} \qquad \text{or} \qquad \Delta_{i,i+1} = \pm\pi - \Delta_{i-1,i}\,.$$

Together with Eq. (2), it implies that the angle vectors are

$$\theta_i^{(q)} = \pi\left[\frac{2q}{n}i - \frac{n-1}{n}q\right]\,, \quad \varphi_i^{(q)} = \pi\left[\frac{2q-1}{n-2}i + \frac{-2n^2k + 2nk - 8qk - n}{2n(n-2)} + \frac{6nq \pm (4nq - n^2)}{2n(n-2)}\right]\,.$$

This gives

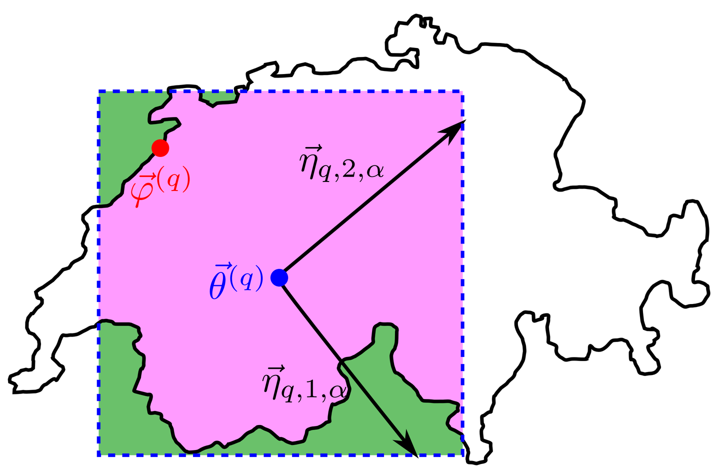
$$\|\vec{\theta}^{(q)} - \vec{\varphi}^{(q)}\|_\infty = \frac{(n-1)(n-4q)}{2(n-2)n}\pi\,,$$

which, for large n, behaves as

$$\|\vec{\theta}^{(q)} - \vec{\varphi}^{(q)}\|_\infty \sim \left(1 - \frac{4q}{n}\right)\,.$$

The volume of the corresponding n-sphere is then

$$V_q \sim \left(1 - \frac{4q}{n}\right)^n \xrightarrow{n\to\infty} \exp(-4q)\,.$$



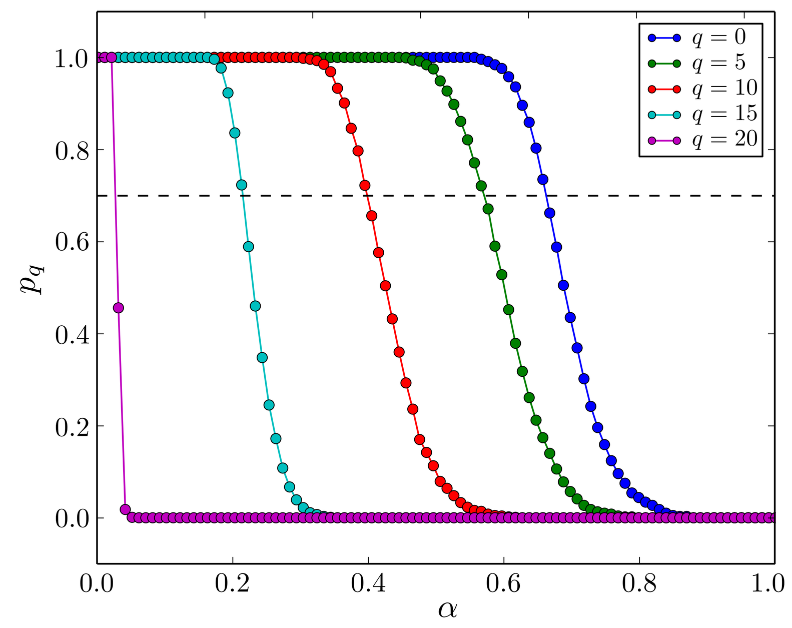
Identical frequencies: Numerical approach

We perform a numerical estimation of the radius of the basin of attraction. We choose $d = 1000$ random **perturbation directions** $\vec{\epsilon}_j$, $j = 1, ..., d$ and define

$$\vec{\eta}_{q,j,\alpha} := \vec{\theta}^{(q)} + \pi\alpha\,\vec{\epsilon}_j\,,$$

where $\alpha \in [0, 1]$ is a tuning parameter. Increasing α, we compute

$$p_q(\alpha) := \frac{1}{d}\text{Card}\left\{\vec{\eta}_{q,j,\alpha} \mid \vec{\theta}(0) = \eta_{q,j,\alpha},\ \vec{\theta}(t\to\infty) = \vec{\theta}^{(q)}\right\}\,.$$



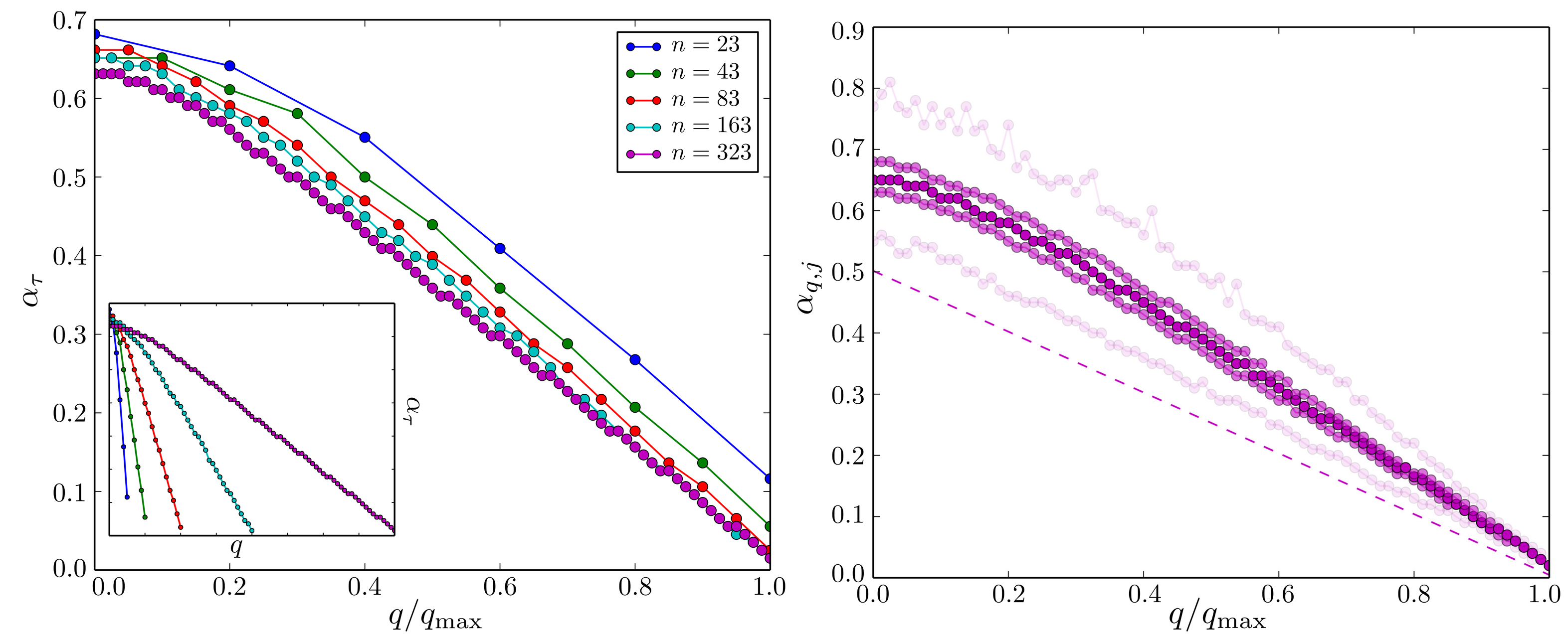
Given a threshold $\tau \in [0, 1]$, we can then define

$$\alpha_\tau(q) := \sup\{\alpha \mid p_q(\alpha) \geq \tau\}\,,$$

as a **typical radius** of the basin of attraction.

The **linear dependence** of the radius of the basin of attraction with respect to the winding number is **confirmed**.

It differs from the Gaussian behavior reported by Wiley et al. [2].



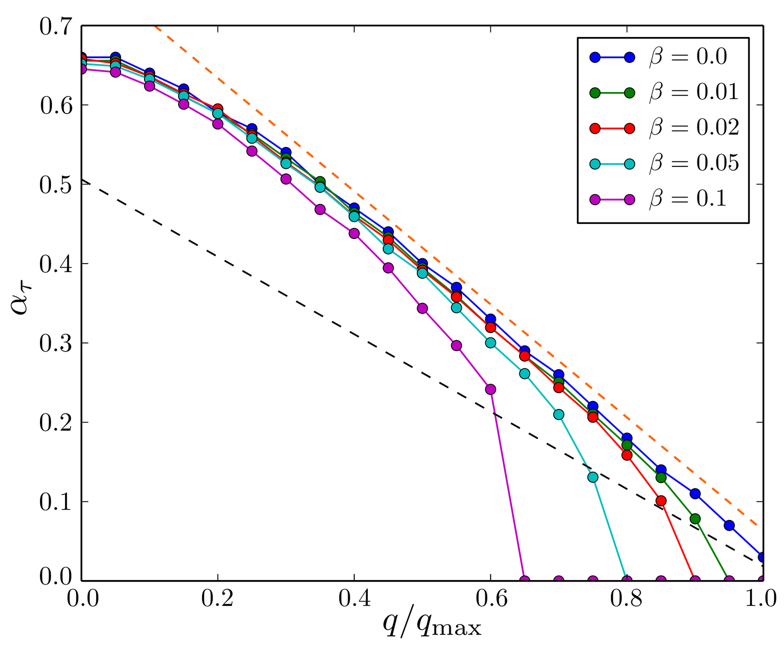
Non-identical frequencies

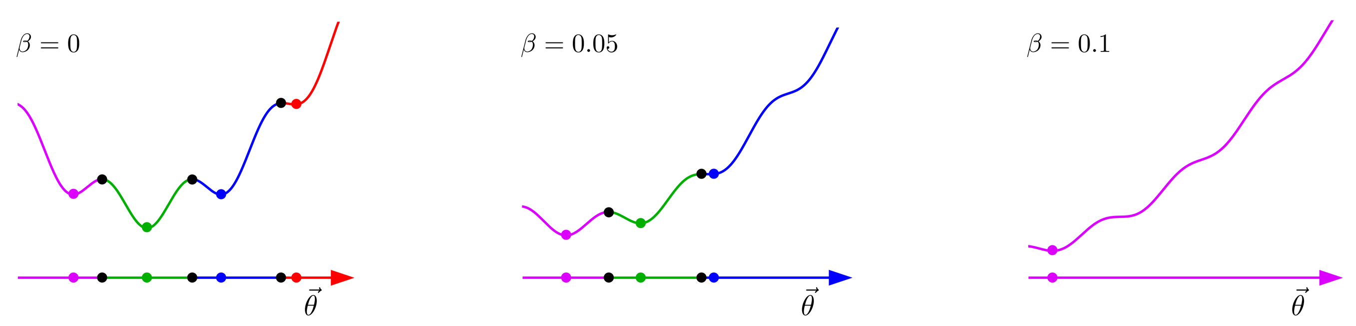
For non-identical frequencies ($P_i \neq 0$), we observe that the radius of the basin of attraction for small winding numbers is preserved. It drops and vanishes for larger winding numbers as the magnitude of the frequencies is increased.

Non-zero frequencies is equivalent to a tilting of the Lyapunov function of the system Eq. (1),

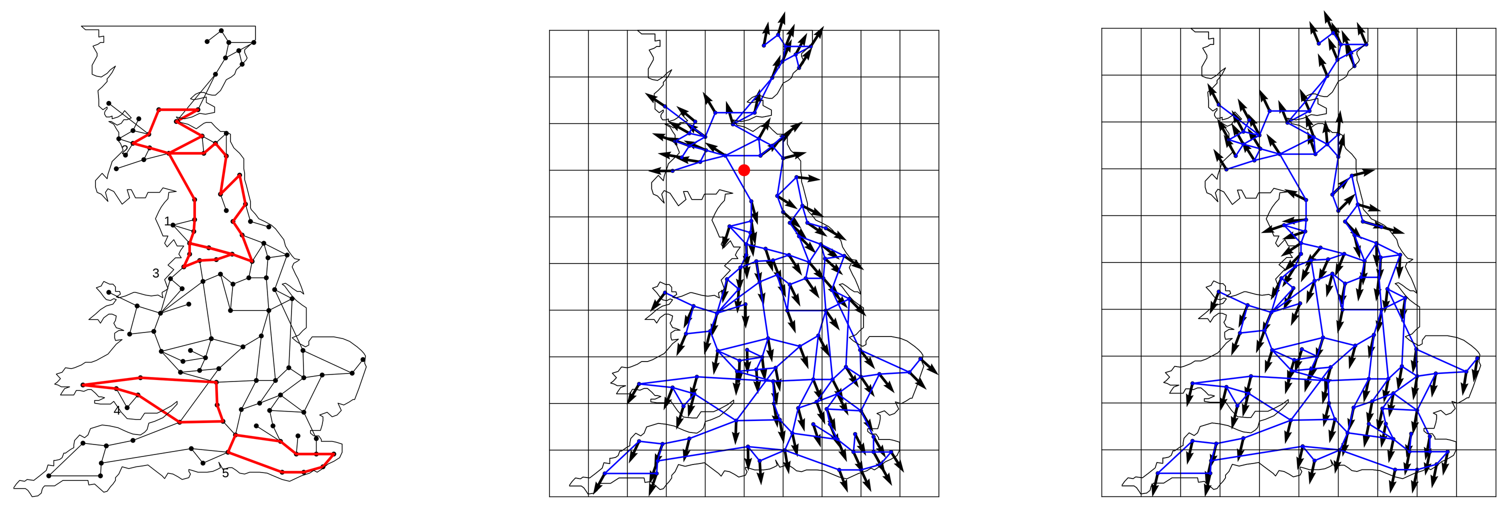
$$\mathcal{V}(\vec{\theta}) = -\sum_i P_i\theta_i - \sum_{i<j} K_{ij}\cos(\theta_i - \theta_j)\,,$$

and the size of the basins of attraction does not change much before it vanishes.





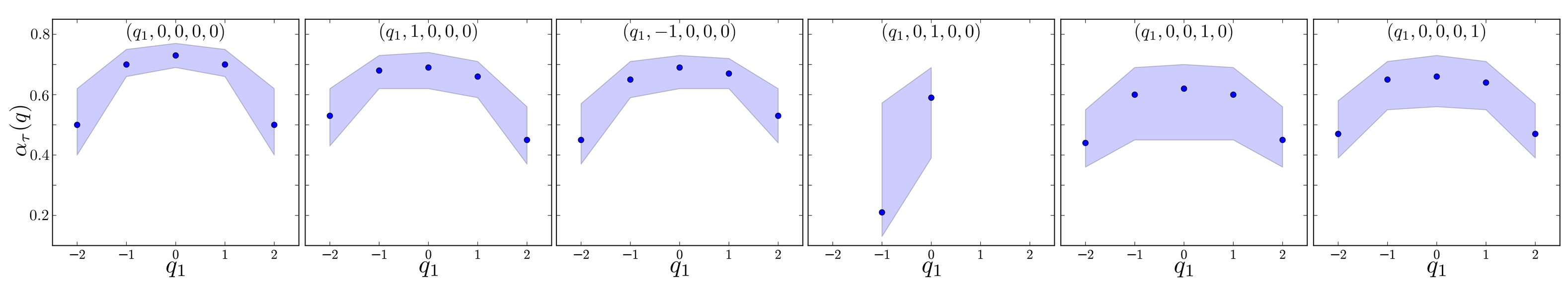
Meshed networks



We estimate the radius of the basins of attraction of the fixed points with various winding numbers on the **large middle cycle** (nr. 1), and fixed winding numbers on the other cycles.

We observe that changing the sign of all winding numbers does not change the radius of the basin of attraction.

The winding number on cycle close to the middle one influences the size of the basins of attraction much more than the winding number on cycles far appart.



References

[1] A. J. Korsak, *IEEE Trans. Power Appar. Syst.* **PAS-91** (1972).
[2] D. A. Wiley, S. H. Strogatz, and M. Girvan, *Chaos* **16** (2006).
[3] R. Delabays, M. Tyloo, and Ph. Jacquod, *Chaos* **27** (2017).

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