Multistability of Phase-Locking and Vortices in Locally Coupled Kuramoto Models

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Introduction

We are interested in the fixed points of the Kuramoto model on a graph G with n vertices and m edges. The dynamics are defined by the system of differential equations

$$\dot{\theta}_i = P_i - K\sum_{j\sim i}\sin(\theta_i - \theta_j)\,, \qquad \text{for } i = 1, ..., n\,, \tag{1}$$

where the sum is taken over neighboring vertices of G and

- $\theta_i \in (-\pi, \pi]$, for $i = 1, ..., n$ are the angles of the oscillators;
- $K \geq 0$ is the coupling constant;
- $P_i \in \mathbb{R}$, for $i = 1, ..., n$, are the natural frequencies of the oscillators, which sum up to zero.

A **fixed point** is then a point on the n-torus, $\vec{\theta} \in \mathbb{T}^n$, satisfying

$$P_i = K\sum_{j\sim i}\sin(\theta_i - \theta_j)\,, \qquad i = 1, ..., n\,. \tag{2}$$

The linear stability of a given fixed point $\vec{\theta}^*$ is obtained by linearizing Eq. (1) around it. Linear stability is thus governed by the **Lyapunov exponents** of the system of Eq. (1), which are the eigenvalues of the **stability matrix**,

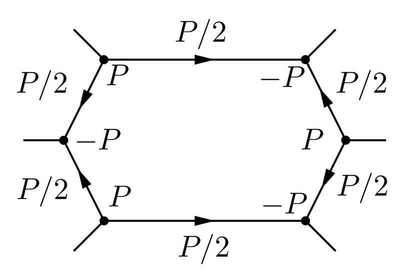
$$M_{ij} \coloneqq \begin{cases} K\cos(\theta_i^* - \theta_j^*) & \text{if } j \neq i \\ -\sum_{k\sim i} K\cos(\theta_i^* - \theta_k^*) & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

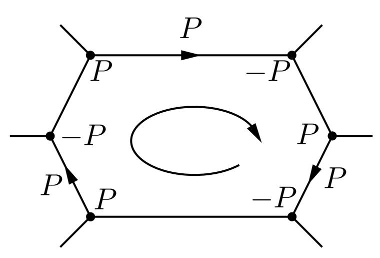
Fixed points are invariant under a uniform rotation of all angles, thus M always has a zero eigenvalue, $\lambda_1 = 0$. The stability condition for a fixed point is then

$$0 = \lambda_1 > \lambda_2 \geq \lambda_2 \geq \cdots \geq \lambda_n\,.$$

Multistability

The Kuramoto model may have multiple stable fixed points depending on the topology of G [1]. Two such fixed points differ by a collection **vortex flows** around the cycles of the network [2, 3].





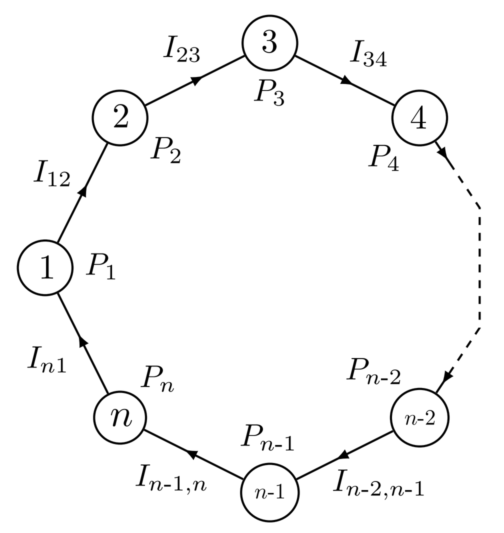
As the angle differences along the edges of the network are taken modulo 2π in the interval $(-\pi, \pi]$, the sum of angle differences around a cycle is an integer multiple of 2π,

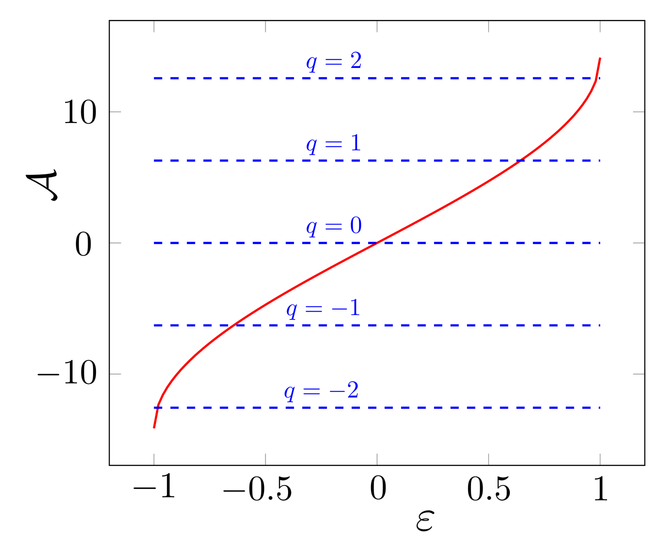
$$\sum_{i=1}^{n_k} |\theta_{i+1} - \theta_i| = 2\pi q_k\,, \qquad q_k \in \mathbb{Z}\,, \tag{3}$$

where the sum is taken over the n_k nodes of any (the k^{th}) cycle of the network, vertex indices are taken modulo n_k and $|\cdot|$ indicates the modulo 2π. The integer q_k is the **winding number** of the stable fixed point on cycle k.

Eq. (3) gives a quantization condition on the angle differences. This implies that there exists a discrete number of vortex flows, and thus a discrete number of stable fixed points.

Number of fixed points in the cycle network





Let us consider a cycle network of length n. If frequencies are identical ($P_i \equiv 0$), Eq. (2) implies that the **flow** on line $\langle i, i+1\rangle$

$$I_{i,i+1} \coloneqq K\sin(\theta_i - \theta_{i+1}) = K\varepsilon\,, \qquad \varepsilon \in [-1, 1]\,, \tag{4}$$

is the same on each line. According to Ref. [4], for a fixed point to be stable, all angle differences have to be in $[-\pi/2, \pi/2]$. Thus Eq. (4) allows us to write the angle differences as $|\theta_{i+1} - \theta_i| = \arcsin(\varepsilon)$, $i = 1, ..., n$. According to Eq. (3), the sum of angle differences around the cycle only takes discrete values,

$$\mathcal{A}(\varepsilon) \coloneqq n\arcsin(\varepsilon) = 2\pi q\,, \qquad q \in \mathbb{Z}\,. \tag{5}$$

As arcsine is an increasing function taking values in $[-\pi, \pi]$, we can determine the number of ε values satisfying Eq. (5), which gives the number $\mathcal{N}$ of stable fixed points of Eq. (1),

$$\mathcal{N} = 2\cdot\text{Int}(n/4) + 1\,. \tag{6}$$

Non-identical frequencies

According to Ref. [4], the number of stable fixed points of Eq. (1) on a cycle is an increasing function of the coupling constant K.

In particular, this implies that if we relax the condition that the frequencies are identical, the right-hand-side of Eq. (6) is an upper bound on the number of stable fixed points,

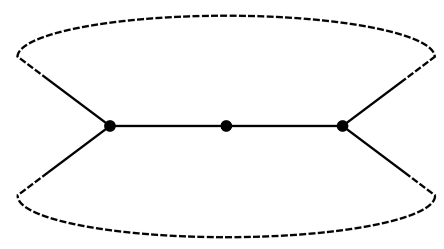
$$\mathcal{N} \leq 2\cdot\text{Int}(n/4) + 1\,. \tag{7}$$

Planar networks

The bound in Eq. (7) can be generalized to a large class of networks [2, 5]. On planar networks, the number of stable fixed points of Eq. (1) with all angle differences in $[-\pi/2, \pi/2]$, denoted $\mathcal{N}^*$, is bounded above by

$$\mathcal{N}^* \leq \prod_{k=1}^{c}[2\cdot\text{Int}(n_k/4) + 1]\,, \tag{8}$$

where c is the number of fundamental cycles of G.



For identical frequencies, if no pair of vertices of degree larger or equal to 3 are conneceted by a single edge, then all the stable fixed point have all angle differences in $[-\pi/2, \pi/2]$ and thus $\mathcal{N} = \mathcal{N}^*$. The right-hand-side of Eq. (8) is then a bound on the total number of stable fixed points [2, 6].

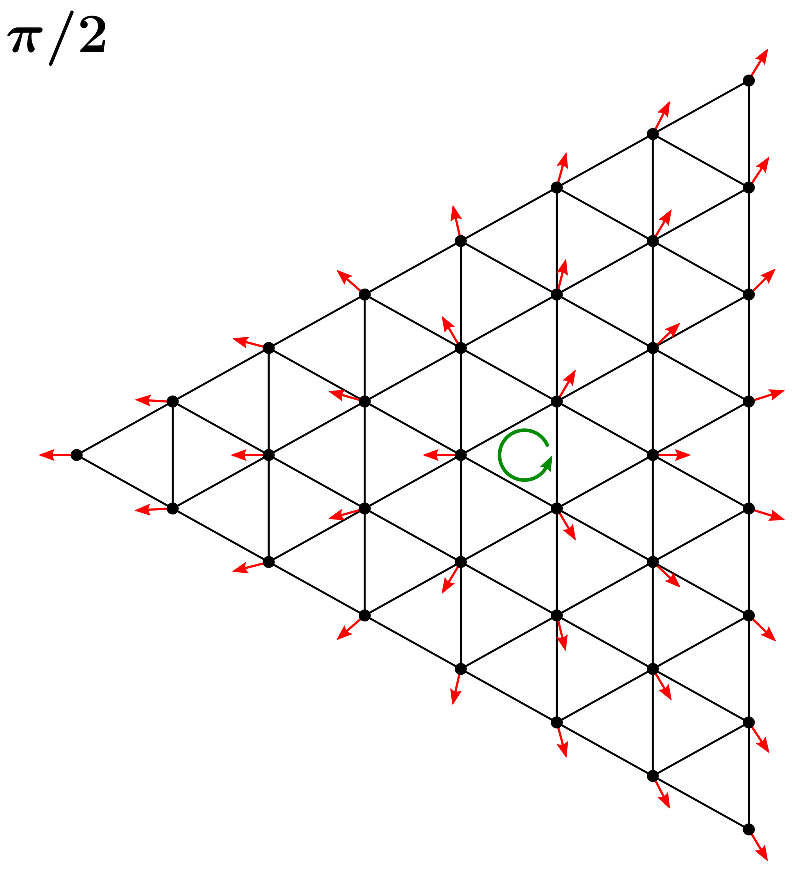
Angle differences exceeding $\pi/2$

If some cycles share a single edge, the right-hand-side of Eq. (8) is not a valid bound any more.

Consider for example the triangular lattice with eight vertices on the boundary, with identical frequencies. All fundamnental cycles have length 3 and thus all terms in the product of Eq. (8) are equal to 1.

Nevertheless, we check numerically that the fixed point having a vortex flow with winding number ± 1 in the center is stable. We then obtain at least three stable fixed points and

$$\prod_{k=1}^{49}[2\cdot\text{Int}(n_k/4) + 1] = 1 < 3 \leq \mathcal{N}\,.$$



Conclusion

We derived an upper bound on the number of stable fixed points of the Kuramoto model on a large class of networks,

$$\mathcal{N} \leq \prod_{k=1}^{c}[2\cdot\text{Int}(n_k/4) + 1]\,.$$

This bound is easily computed from the topology of the network considered.

Furthermore, this bound is algebraic in the length of the cycles of the network. This significantly improves previously known bounds which are exponential in the number of vertices of the network.

Despite the wide class of network topologies covered by our bound, some particular cases violate it, as we pointed out by an example. This implies that our bound cannot be extended to all network topologies.

References

[1] A. J. Korsak. On the question of uniqueness of stable load-flow solutions. *IEEE Trans. Power App. Syst.*, PAS-91(3):1093–1100, 1972.

[2] R. Delabays, T. Coletta, and Ph. Jacquod. *arXiv preprint: 1609.02359*, 2016.

[3] F. Dörfler, M. Chertkov, and F. Bullo. Synchronization in complex oscillator networks and smart grids. *Proc. Natl. Acad. Sci.*, 110(6):2005–2010, February 2013.

[4] R. Delabays, T. Coletta, and Ph. Jacquod. *J. Math. Phys.*, 57(3), 2016.

[5] D. Manik, M. Timme, and D. Witthaut. *arXiv preprint arXiv:1611.09825*, 2016.

[6] R. Taylor. There is no non-zero stable fixed point for dense networks in the homogeneous Kuramoto model. *J. Phys. A*, 45(5):055102, 2012.

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