Loop Flows and the Number of Power Flow Solutions in Meshed Electric Power Grids

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Joint work with

T. Coletta and P. Jacquod

R. Delabays, T. Coletta, and P. Jacquod, *J. Math. Phys.* **57** (2016)
R. Delabays, T. Coletta, and P. Jacquod, *to appear in J. Math. Phys.* (2017)

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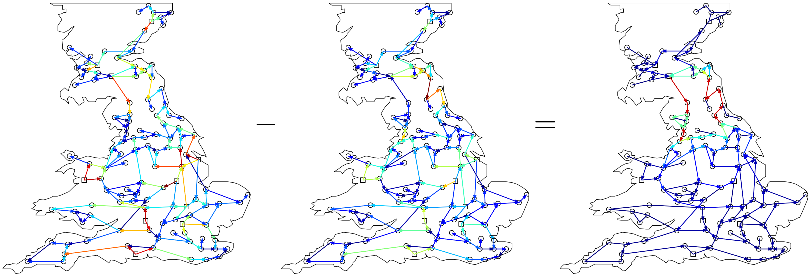
Multistability and Loop Flows

Theorem: *The Lossless AC Power Flows on meshed networks may have multiple stable fixed points whose differences are collections of* **Loop Flows**.



Coletta *et al.*, *New J. Phys.* **18** (2016)

F. Dörfler, M. Chertkov, and F. Bullo, *Proc. Natl. Acad. Sci.* **110** (2013)

R. Delabays, T. Coletta, and P. Jacquod, *J. Math. Phys.* **57** (2016)

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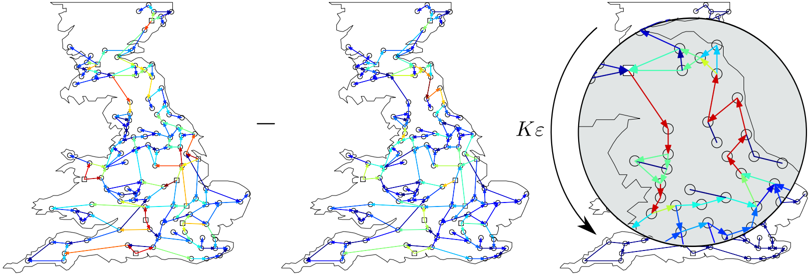

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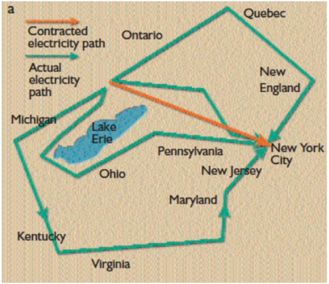
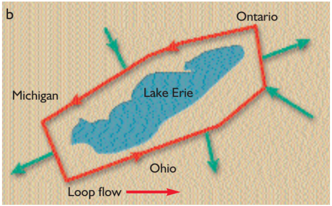
Multistability and Loop Flows





How many?

J. Casazza, *Electrical World* (1998)

E. J. Lerner, *The Industrial Physicist* **9** (2003)

The Starting Point

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The **Lossless AC Power Flow Equations**: $\forall i = 1, \ldots, n$

$$P_i = \sum_{j=1}^{n} \underbrace{|V_i||V_j|B_{ij}}_{K} \sin(\theta_i - \theta_j)\,,$$

$$G = 0 \quad \text{and} \quad |V_i| \equiv V\,,$$

$$\text{Consider identical coupling: } K := V^2 B_{ij}\,,$$

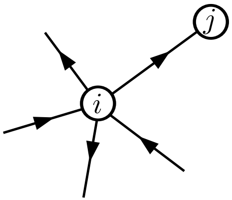
where

- P_i: active power at bus i;
- $|V_i|$, θ_i: voltage amplitude and phase at bus i;
- B_{ij}: susceptance of the line between buses i and j.

The Starting Point

We end up with: $$P_i = \sum_{j\sim i} K \sin(\theta_i - \theta_j)\,,$$

P_i, K: parameters,
θ_i: unknowns.



Fixed point of the **Swing Equations**:

$$\dot{\theta}_i = P_i - \sum_{j\sim i} K \sin(\theta_i - \theta_j)\,,$$

with identical damping and no intertia.

Angles differences: $\boxed{<\pi/2}$ or $\boxed{>\pi/2}$ $\implies$ $\mathcal{N} \sim 2^{\#\text{edges}}$.

D. Mehta, H. Nguyen, and K. Turistyn, *IET Gen. Trans. & Dist.* **10** (2016)

Stability

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A fixed point $\{\theta_i^*\}$ of $\quad \dot{\theta}_i = P_i - \sum_{j\sim i} K \sin(\theta_i - \theta_j)\,,$

is **linearly stable** if and only if the **stability matrix** M defined as

$$M_{ij} := \begin{cases} -\sum_{k\sim i} K\cos(\theta_i^* - \theta_k^*)\,, & \text{if } i = j\,, \\ K\cos(\theta_i^* - \theta_j^*)\,, & \text{if } i \sim j\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

is **negative semi-definite**.

$$0 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n\,.$$

Note: inertia does not discard stability.

D. Manik *et al.*, *Eur. Phys. J.* **223** (2014)

T. Coletta and P. Jacquod, *Phys. Rev. E* **93** (2016)

Discretization of loop flows

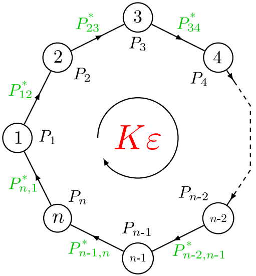
$$P_{ij} = P_{ij}^* + K\varepsilon := K\sin(\theta_i - \theta_j)$$

$$\iff |\theta_i - \theta_j|_{2\pi} = \arcsin(P_{ij}^*/K + \varepsilon)\,.$$

Voltage angles are uniquely defined:

$$V_j = Ve^{i\theta_j}\,,$$

$\rightarrow$ **Winding Number**: $q_k \in \mathbb{Z}$



$$\mathcal{A}(\varepsilon) := \sum_{i=1}^{n_k} |\theta_i - \theta_{i+1}|_{2\pi} = \sum_{i=1}^{n_k} \arcsin(P_{i,i+1}^*/K + \varepsilon) = 2\pi q_k\,.$$

Discretization of loop flows

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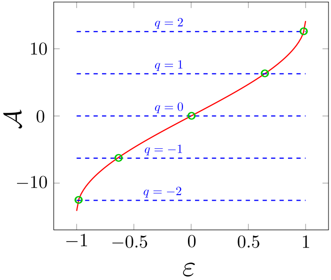


$$\mathcal{A}(\varepsilon) := \sum_{i=1}^{n_k} |\theta_i - \theta_{i+1}|_{2\pi} = \sum_{i=1}^{n_k} \arcsin(P^*_{i,i+1}/K + \varepsilon) = 2\pi q_k\,.$$

$\implies$ There is a **discrete** number of possible loop flows.

But how many?

One cycle - $P \equiv 0$ (i.e. $K \to \infty$)

Identical flow on the lines:

$$-1 \leq \varepsilon \leq 1\,.$$

Identical angle differences on the lines:

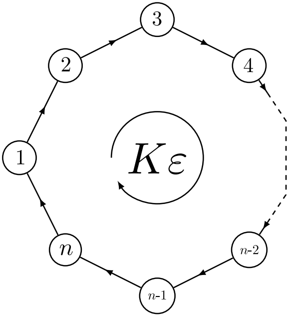
$$\boxed{-\pi/2 \leq \arcsin(\varepsilon) \leq \pi/2\,.}$$

Discretization:

$$n \cdot \arcsin(\varepsilon) = 2\pi q\,, \qquad q \in \mathbb{Z}\,,$$

$$\implies q \in \{-\mathrm{Int}(n/4), ..., \mathrm{Int}(n/4)\}\,,$$

$$\implies \boxed{\mathcal{N} = 2 \cdot \mathrm{Int}(n/4) + 1} \qquad \left(\neq 2 \cdot \mathrm{Int}\left((n-1)/4\right) + 1\right).$$



R. Delabays, T. Coletta, and P. Jacquod, *J. Math. Phys.* **57** (2016)

D. Manik, M. Timme, and D. Witthaut, arXiv 1611.09825 (2017)

One cycle - $P \equiv 0$ (i.e. $K \to \infty$)

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$$\boxed{-\pi/2 \leq |\theta_i - \theta_{i+1}|_{2\pi} \leq \pi/2}$$

$$|\theta_i - \theta_{i+1}|_{2\pi} \in \{\arcsin(\varepsilon), \pi - \arcsin(\varepsilon)\} \implies \cos(\theta_i - \theta_{i+1}) = \pm c$$

If $|\theta_{i-1} - \theta_i|_{2\pi} = \pi - \arcsin(\varepsilon)$ and $|\theta_i - \theta_{i+1}|_{2\pi} = \arcsin(\varepsilon)$,

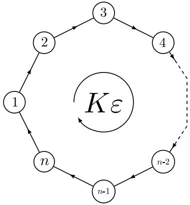
$$-M = K \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & x & c & 0 & \\ & c & 0 & -c & \\ & 0 & -c & y & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \qquad \begin{vmatrix} x & c \\ c & 0 \end{vmatrix} = -c^2,$$

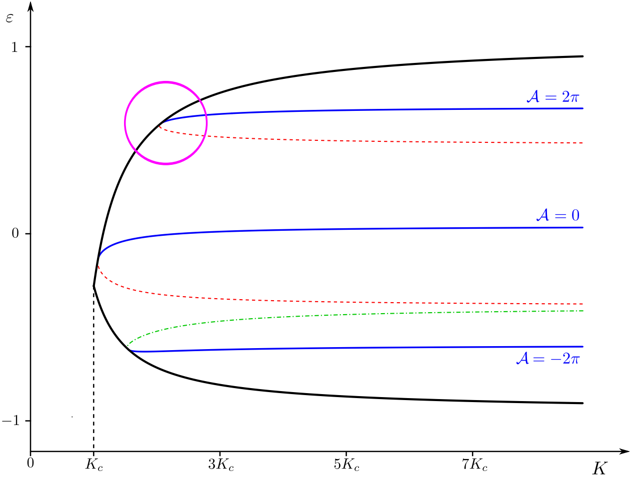
negative principal minor

Sylvester's criterion $\implies$ M is not NSD $\implies$ unstable ($\lambda_2 > 0$).

One cycle - $P \neq 0$ (i.e. $K < \infty$)

Decrease $K \implies \mathcal{N} \leq 2 \cdot \mathrm{Int}(n/4) + 1$





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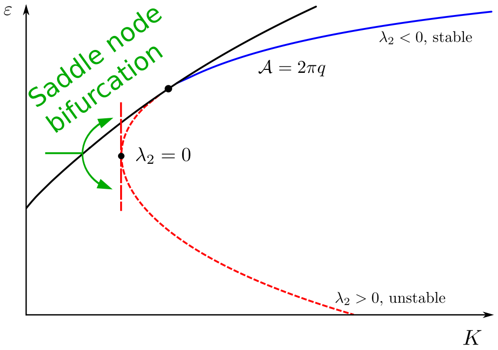
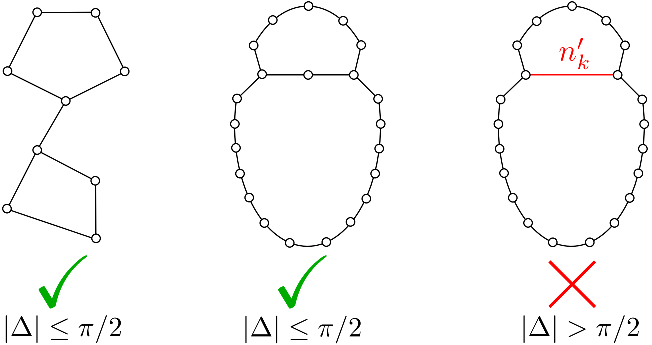


R. Delabays, T. Coletta, and P. Jacquod, *J. Math. Phys.* **57** (2016)

Planar networks - $P \equiv 0$

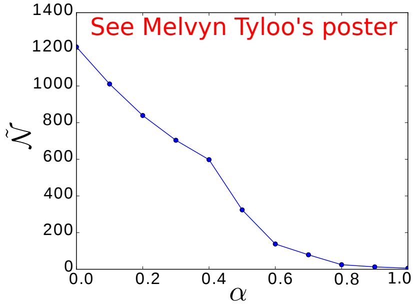
$$\text{Conjecture: } \mathcal{N} \leq \prod_{k=1}^{c} \left[2 \cdot \mathrm{Int}\big((n_k + n'_k)/4\big) + 1\right] .$$

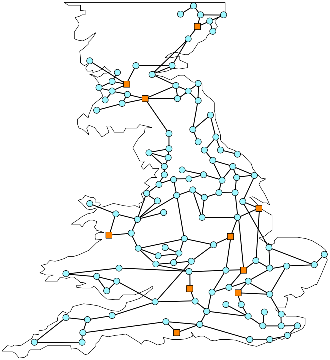


R. Delabays, T. Coletta, and P. Jacquod, *to appear in J. Math. Phys.* (2017)

Planar networks

120 nodes, 165 lines, 45 cycles, $K = 13$
10 producers: $P = +11\alpha$,
110 consumers: $P = -\alpha$





$$\prod_{k=1}^{c} \left[2 \cdot \mathrm{Int}\left((n_k + n'_k)/4\right) + 1\right] \approx 1.475 \cdot 10^{27}$$

Conclusion

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Conclusions:

- Multistability occurs in the models used to describe the electrical grid;
- Fixed points differ by a collection of Loop Flows;
- $\mathcal{N} \leq \prod_{k=1}^{c} \left[2 \cdot \mathrm{Int}\left(n_k/4\right) + 1\right]$.

Further questions:

- Non-zero injections on planar graph: No proof yet!
- Consider dissipation;
- Can we observe non-zero winding numbers in real-life data?

Thank you!