Multistability of Phase-Locking and Topological Winding Numbers in Locally Coupled Kuramoto Models

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Reference: R. Delabays, T. Coletta, and P. Jacquod
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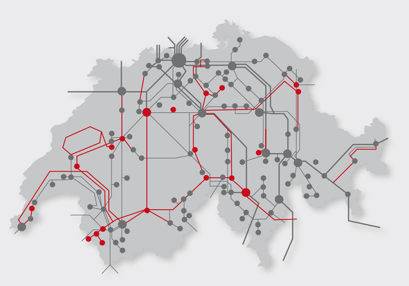
Motivation

Electrical network



Source: Swissgrid

A **graph** with:

- n **vertices (buses)** characterized by a voltage: $|V_j|e^{i\theta_j}$,
- m **edges (lines)** with admittance: $Y_{jk} = G_{jk} + iB_{jk}$.

The model

Power Flow Equations: for all $i = 1, ..., n$

$$P_i = \sum_{j=1}^{n} |V_i||V_j| \left[B_{ij} \sin(\theta_i - \theta_j) + G_{ij} \cos(\theta_i - \theta_j)\right] \ ,$$

$$Q_i = \sum_{j=1}^{n} |V_i||V_j| \left[G_{ij} \sin(\theta_i - \theta_j) - B_{ij} \cos(\theta_i - \theta_j)\right] \ .$$

High voltage networks, $\forall i, j$:

$$G_{ij} \approx 0 \quad \text{and} \quad |V_i| \approx |V| \, .$$

Other assumption, $\forall \langle i, j \rangle$:

$$K_{ij} := |V|^2 B_{ij} = K \ .$$

A. R. Bergen and V. Vittal, *Power Systems Analysis* (2000)

The model

Σ π ≈ &

We obtain the following reduced **Power Flow Equations**:

$$P_i = \sum_{j \sim i} K \sin(\theta_i - \theta_j) \quad \forall i.$$

We denote the **transmitted power** on line $\langle ij \rangle$:

$$I_{\langle ij \rangle} = K \sin(\theta_i - \theta_j) = K \sin(\Delta_{ij}), \quad \Delta_{ij} \in (-\pi, \pi]$$

Kirchhoff's Current Law (**KCL**):

$$P_i = \sum_{j \sim i} I_{\langle ij \rangle}$$

Stability

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Dynamics are given by the **Swing Equations**, which reduce here to the **Kuramoto model**:

$$\dot{\theta}_i = P_i - \sum_{j \sim i} K \sin(\theta_i - \theta_j) \ .$$

For a given solution $\{\theta_i^{(0)}\}$, linear stability is given by the eigenvalues of the **stability matrix**:

$$M_{ij} := \begin{cases} K \cos(\theta_i^{(0)} - \theta_j^{(0)}) \, , & \text{if } i \neq j \, , \\ -\sum_{k \sim i} K \cos(\theta_i^{(0)} - \theta_k^{(0)}) \, , & \text{if } i = j \, . \end{cases}$$

Remark: $\lambda_1 = 0$ and $\lambda_2 > ... > \lambda_n$. A solution is stable if and only if $\lambda_2 \leq 0$.

Two solutions differ by loop flows

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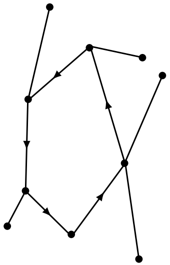
Theorem

Let G be a graph and $P \in \mathbb{R}^n$ a vector of power injections and consumptions. Two flow repartitions $I', I'' \in \mathbb{R}^m$ satisfying KCL differ by a collection of loop flows.



Proof.

Let $A \in \mathbb{R}^{n\times m}$ be the incidence matrix of G,

$$P = AI' = AI''\,.$$

Thus $(I' - I'') \in \ker(A)$, which is generated by the cycles of G. □

F. Dörfler, M. Chertkov, and F. Bullo, *Proc. Natl. Acad. Sci.* **110** (2013)
R. Delabays, T. Coletta, and P. Jacquod, *J. Math. Phys.* **57** (2016)

Example

Let G be a tree,

$$\implies \quad \exists! \quad I \in \mathbb{R}^m \text{ st. } AI = P\,,$$

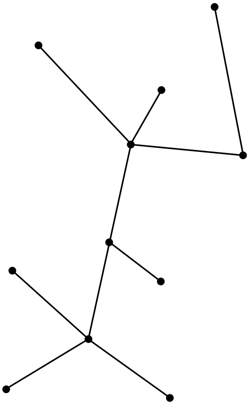
and for any edge $\langle ij \rangle$,

$$\Delta_{ij} = \arcsin(I_{\langle ij \rangle}/K)$$

or

$$\Delta_{ij} = \pi - \arcsin(I_{\langle ij \rangle}/K)\,.$$

Implying, **a priori**, 2^{n-1} solutions.



Taylor's Proposition

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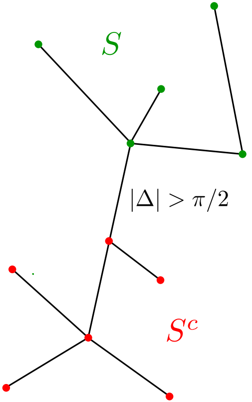
Proposition (Taylor, 2012)

Let $\{\theta_i^{(0)}\}$ be any stable solution of the Kuramoto model on G. Then for any non-empty vertices subset S,

$$\sum_{\langle ij \rangle \,:\, i \in S, j \notin S} \cos(\Delta_{ij}^{(0)}) \geq 0 \,.$$

Then if G is a tree, there is a **unique** stable solution.



R. Taylor, *J. Phys. A* **45** (2012)

One cycle

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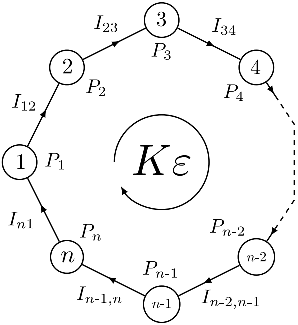
Let $G = \mathcal{C}_n$. For $i = 1, \ldots, n$, define

$$I^*_{\langle i,i+1\rangle} := \sum_{j=1}^{i} P_j \,.$$



Any flow repartition $I \in \mathbb{R}^m$ can be written

$$I_{\langle i,i+1\rangle} = I^*_{\langle i,i+1\rangle} + K\varepsilon \,.$$

And the corresponding angle differences

$$\Delta_{i,i+1} = \begin{cases} \arcsin\left(I^*_{\langle i,i+1\rangle}/K + \varepsilon\right) \,, \\ \pi - \arcsin\left(I^*_{\langle i,i+1\rangle}/K + \varepsilon\right) \,. \end{cases}$$

Angle differences

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According to Taylor's Proposition there is at most **one** angle difference

$$|\Delta_{\langle i,i+1\rangle}| > \pi/2\,.$$

One can show (Delabays, Coletta, and Jacquod, 2016) that for $K \to \infty$ (or $P \to 0$), stable solutions have all

$$\Delta_{\langle i,i+1\rangle} \in [-\pi/2, \pi/2]\,.$$

R. Delabays, T. Coletta, and P. Jacquod, *J. Math. Phys.* **57** (2016)

Sum of angle differences

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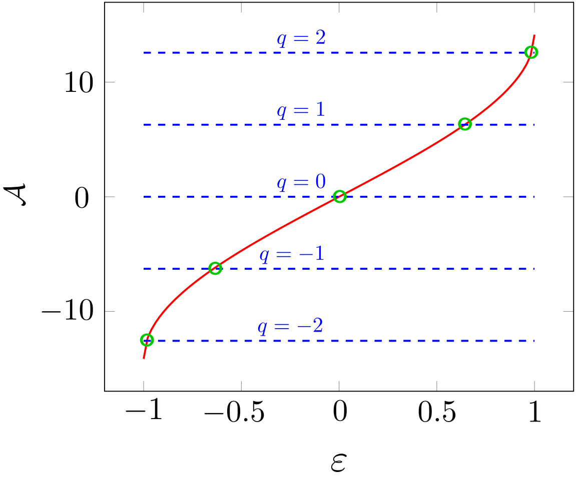
Considering $\Delta_{i,i+1} \in [-\pi/2, \pi/2]$, the sum of angle differences around the cycle reads:

$$\begin{aligned}\mathcal{A}(\varepsilon) &= \sum_{i=1}^{n} \Delta_{i,i+1} \\ &= \sum_{i=1}^{n} \arcsin\left(I^*_{\langle i,i+1\rangle}/K + \varepsilon\right) \\ &\stackrel{!}{=} 2\pi q\,, \quad q \in \mathbb{Z}\,.\end{aligned}$$

Any solution is then characterized by a topological **winding number**, $q \in \mathbb{Z}$.

Multiple solutions, cycle of length 9



Bound on the number of solutions

For $K \to \infty$,

$$\mathcal{A}(\varepsilon) \to n \arcsin(\varepsilon), \quad \varepsilon \in [-1, 1]$$

$\mathcal{A}$ takes value in $[-n\pi/2, n\pi/2]$.

Each multiple of 2π gives a solution.

The number of solutions is

$$\mathcal{N}_\infty = 2 \cdot \mathrm{Int}(n/4) + 1 \,.$$

J. A. Rogge and D. Aeyels, *J. Phys. A* **37** (2004)
J. Ochab and P. F. Góra, *Acta Phys. Pol. B Proc. Suppl.* **3** (2010)

To conclude

Σ π ≈ &

Theorem (Delabays, Coletta, and Jacquod, 2016)

The number $\mathcal{N}$ of stable solutions of the Kuramoto model on a cycle is an increasing function of K.

Corollary

The number of stable solutions is bounded by

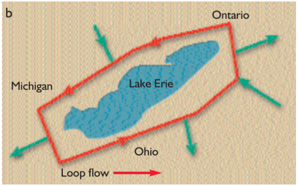
$$\mathcal{N} \leq 2 \cdot \mathrm{Int}(n/4) + 1 \,.$$

Remark: When K decreases, some solutions may have one angle difference $|\Delta| > \pi/2$.

R. Delabays, T. Coletta, and P. Jacquod, *J. Math. Phys.* **57** (2016)

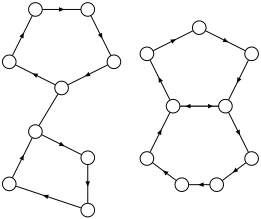
Conclusion

- Real-life networks (e.g. Lake Erie);
- Unnecessary losses.



Next steps:

- Loop flow generation;
- Multiple cycles.



E. J. Lerner, *The Industrial Physicist* **9** (2003)